BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 5, September 1973

THE NORMALITY OF A PRODUCT WITH A COMPACT FACTOR

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Communicated January, 1973

Assume all spaces are T_1 and all maps continuous.

Suppose X is compact and Y normal. Even if X is the closed unit interval there is no reason to expect $X \times Y$ to be normal [**1**]. Classically we have: $X \times Y$ is normal if Y is paracompact [**2**]; $\beta Y \times Y$ is normal only if Y is paracompact [**3**]; and $[0, 1] \times Y$ is normal if and only if Y is countably paracompact [**4**]. The purpose of this paper is to announce that the following conjecture of Morita [**5**], [**6**] is true.

THEOREM. *If X is compact, $X \times Z$ normal, and Y the image of Z under a closed map, then $X \times Y$ is normal.*

Partial results have been obtained by T. and K. Chiba.

To show the spirit of this proof we now make some definitions and then give the principal lemmas of the proof.

If κ is a cardinal, we say the space Z is *κ-collectionwise normal* provided, for all discrete families $\{F_\alpha\}_{\alpha<\kappa}$ of subsets of Z, there is a family $\{U_\alpha\}_{\alpha<\kappa}$ of disjoint open sets with $F_\alpha \subset U_\alpha$ for each $\alpha < \kappa$. We say a family $\{G_\alpha\}_{\alpha<\kappa}$ of subsets of a space Y is *hereditarily closure preserving* provided each family $\{J_\alpha\}_{\alpha<\kappa}$, with $J_\alpha \subset G_\alpha$ for each α, is closure preserving in Y. Suppose X and Z are spaces and $g:(X \times Z) \to [-1, 1]$ is continuous. For $z \in Z$ define $U_{z,g} = \{u \in Z \mid \text{for all } x \in X,\ g(x, u) = -1 \text{ implies } g(x, z) < 0 \text{ and } g(x, u) = 1 \text{ implies } g(x, z) > 0\}$. Then we call $\{U_{z,g} \mid z \in Z\}$ a *separating set for Z with respect to X.*

Lemma 1 in a way attaches a cardinal function to the fact that $X \times Z$ is normal for a compact X:

LEMMA 1. *Suppose X is compact, $X \times Z$ is normal, and $\mathcal{U}$ is a separating set for Z with respect to X. Then there is a cardinal κ such that Z is κ-collectionwise normal and $\mathcal{U}$ has a locally finite closed refinement of cardinality κ.*

Observe that closed maps preserve κ-collectionwise normality and take locally finite families into hereditarily closure preserving families. Thus

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AMS (MOS) subject classifications (1970). Primary 54B10, 54C10, 54A25; Secondary 54D20, 54D15.

Key words and phrases. Product, normality, compact, closed map.

[1] Partially supported by NSF GP-29537X.

it is easy to prove that Lemma 2 completes the proof of Morita's conjecture. However I feel Lemma 2, which does not mention products, may be useful in many situations.

LEMMA 2. *If Y is κ-collectionwise normal and an open cover of Y has a cardinality κ hereditarily closure preserving closed refinement, then it has a locally finite closed refinement.*

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