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SKEW-PRODUCT FLOWS, FINITE EXTENSIONS OF MINIMAL TRANSFORMATION GROUPS AND ALMOST PERIODIC DIFFERENTIAL EQUATIONS[1]

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I. **Skew-product flows.** A flow π on a product space $X \times Y$ is said to be a *skew-product flow* if there exist continuous mappings $\varphi: X \times Y \times T \to X$ and $\sigma: Y \times T \to Y$ such that

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t))$$

where σ is itself a flow on Y and T is a topological group. In other words the natural projection $p: X \times Y \to Y$ is a homomorphism of the transformation group $(X \times Y, T, \pi)$ onto (Y, T, σ).

Skew-product flows arise in a natural way in the study of ordinary differential equations $x' = g(x, t)$ (cf. [**6**] and [**7**]). In this case the group T would be the real numbers and Y would be a topological function space containing g and closed under time-translations. The flow σ would be given by $\sigma(f, \tau) = f_\tau$ where $f_\tau(x, t) = f(x, \tau + t)$. The space X would be the phase space for the differential equation, usually X is the Euclidean space R^n or perhaps some n-dimensional manifold, and $\varphi(x, f, t)$ would represent the solution of $x' = f(x, t)$ passing through x at time $t = 0$. (We assume that all differential equations in Y give rise to unique solutions, although some of our results are valid without this restriction (cf. [**8**]).)

Now assume that Y is a compact minimal set under the flow σ and let $M \subset X \times Y$ be a compact invariant set of the skew-product flow. Motivated by the above model for differential equations we ask: When can certain structures be lifted from Y to M? For example, if we assume that Y is an almost periodic minimal set (that is, the flow σ is equicontinuous on Y) under what conditions will M contain an almost periodic minimal set?

We shall say that the flow π has the *distal property* on M if for any $y \in Y$ and $x_1, x_2 \in X$ with $x_1 \neq x_2$, $(x_1, y) \in M$ and $(x_2, y) \in M$ there is an

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$\alpha = \alpha(x_1, x_2, y) > 0$ such that $d(\varphi(x_1, y, t), \varphi(x_2, y, t)) \geqq \alpha$ for all $t \in R^+$. Here d denotes a metric on X. (For our purposes the R^+ above may be replaced by R^-.)

We can now prove the following theorem [**4**], [**5**]:

THEOREM 1. *Assume that Y is a compact uniform Hausdorff space and the flow σ is minimal on Y. Assume that X is metrizable and $T = R$. Let $M \subset X \times Y$ be a compact invariant set for the flow π and assume either*:

(I) card$(p^{-1}(y) \cap M) = N < \infty$ *for all $y \in Y$, where N does not depend on y, or*

(II) card$(p^{-1}(y_0) \cap M) = N < \infty$ *for some $y_0 \in Y$ and π has the distal property on M.*

Then M is an N-fold covering space of Y. Also M can be written as the finite union of minimal sets. If, in addition, Y is almost periodic minimal then every minimal set in M is also almost periodic.

The assumption that X be metrizable (and not merely a uniform space) is used in a crucial way in our proof. The fact that Y can be a nonmetrizable uniform space does arise in differential equations when Y has a weak topology. In the case that both X and Y are metrizable then Theorem 1 is a consequence of a more general result which we now describe.

II. **Finite extensions of minimal transformation groups.** Recall that a continuous mapping p of a transformation group (W, T, π) onto a transformation group (Y, T, σ) is said to be a *homomorphism* if p commutes with t, that is, if $\sigma(p(w), t) = p(\pi(w, t))$. Also p is said to be a homomorphism of *distal type* if whenever $w_1, w_2 \in p^{-1}(y)$ with $w_1 \neq w_2$, there is an $\alpha = \alpha(w_1, w_2) > 0$ such that $d(\pi(w_1, t), \pi(w_2, t)) \geqq \alpha$ for all $t \in T$. The space W is said to be a *finite* (N-*to*-1) *extension* of Y if card $p^{-1}(y) = N < \infty$ for all $y \in Y$.

The next result places no restriction on the topological group T.

THEOREM 2. *Let W and Y be compact metric spaces where the flow σ on Y is minimal. Let $p: W \to Y$ be a homomorphism. Then the following statements are equivalent*:

(I) *W is a finite* (N-*to*-1) *extension of Y.*

(II) *p is of distal type and* card $p^{-1}(y_0) = N$ *for some $y_0 \in Y$.*

(III) *W is an N-fold covering space of Y with covering projection p.*

In [**2**, p. 56], R. Ellis asks whether an equicontinuous structure on Y can be lifted to a finite (N-to-1) extension of Y. We can give an affirmative answer, but now we must place a rather mild restriction on the group T.

THEOREM 3. *Let $p: W \to Y$ be a homomorphism where W and Y are compact metric spaces. Assume the following*:

(I) (Y, T, σ) *is equicontinuous.*

(II) W *is an* N*-fold covering space of* Y *with covering projection* p.

(III) *The group* T *has the property that there is a compact subset* $K \subset T$ *such that* T *is generated by any open neighborhood of* K.

Then (W, T, π) *is equicontinuous.*

The class $\mathscr{T}$ of topological groups that satisfy condition (III) above is very large. $\mathscr{T}$ contains all compactly generated groups, all connected groups, and $\mathscr{T}$ is closed under arbitrary products with the standard product topology. However, $\mathscr{T}$ does not include infinitely generated discrete groups.

III. **Almost periodic differential equations.** Let us now return to the differential equation model described in §I, where we now assume that Y is an almost periodic minimal set. This means that Y is the hull $H(g)$ generated by a differential equation $x' = g(x, t)$ where g is uniformly Bohr almost periodic in t (cf. [7]). The problem of determining whether a set $M \subset X \times Y$ contains an almost periodic minimal set is the same as asking whether the given differential equation $x' = g(x, t)$ has an almost periodic solution (cf. [7]). If $x' = g(x, t)$ has a positively compact solution $\varphi(x, g, t)$, that is, φ remains in a compact set for $t \geqq 0$, then the ω-limit set $M = \Omega_{(x,g)}$ is a compact invariant set in $X \times Y$. If the positively compact solution $\varphi(x, g, t)$ is uniformly stable [7] then we can show that the solutions have the distal property on M, and that M is a minimal set. For an application of Theorem 1, it remains only to check the finiteness condition card $(p^{-1}(y_0) \cap M) = N < \infty$ for some $y_0 \in Y$. However, if the positively compact solution $\varphi(x, g, t)$ is uniformly asymptotically stable then we can verify this finiteness condition; and hence M is an N-fold covering of Y and there exists an almost periodic solution of $x' = g(x, t)$. Thus the theorems of R. K. Miller [3] and T. Yoshizawa [9] are special cases of Theorem 1.

The theory of L. Amerio [1] is also included in Theorem 1. He assumed a separatedness condition which is much stronger than the distal property used in Theorem 1. This separatedness condition already implies the finiteness condition card $(p^{-1}(y_0) \cap M) = N < \infty$.

For the scalar-valued differential equation $x' = g(x, t)$ we can prove the following result.

THEOREM 4. *Let* $x' = g(x, t)$ *be a scalar-valued differential equation where* g *is uniformly Bohr almost periodic in* t. *If there exists a positively bounded uniformly stable solution* $\varphi(x, g, t)$, *then the* ω*-limit set* $M = \Omega_{(x,t)}$ *is a* 1*-cover of* Y *and* M *is an almost periodic minimal set.*

This result is interesting because we are able to drop the asymptotic stability assumption which Miller and Yoshizawa used in their theories.

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