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EXISTENCE OF THE STABLE HOMOTOPY FAMILY $\{\gamma_t\}$

BY RAPHAEL ZAHLER

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Toda [7] has asked whether Smith's $V(n)$-construction for $n = 3$ yields a nontrivial element γ_1 in the p-component ${}_p\pi_*^S$ of the stable homotopy of spheres (p a prime, $p \geqq 5$).[1] This question has become a major stumbling block, since γ_1 has stubbornly refused to be detected by most conventional invariants [9]. We can now show that γ_1 is essential; moreover (for $p \geqq 7$) it is only the first of a new family $\{\gamma_t\}$ of stable homotopy elements, which are nontrivial for $t \leqq p - 1$ at least. The family $\{\gamma_t\}$ parallels the known infinite families $\{\alpha_t\}$ and $\{\beta_t\}$ ([1], [4], [8], [10], [12]).

We define γ_t to be the composite

$$S^{2t(p^3-1)} \hookrightarrow S^{2t(p^3-1)}V(2) \xrightarrow{\chi_t} V(2) \to S^{2p^2+2p-1},$$

in the stable category, where the $V(n)$ are the spectra introduced by Smith ([6], [8]), $\chi: S^{2(p^3-1)}V(2) \to V(2)$ is a map whose cone is $V(3)$, and χ_t is the usual iterate of suspensions of χ. The map χ is known to exist only for $p \geqq 7$, but a similar construction defines γ_1 for $p = 5$ as well [7].

THEOREM A. *The element* $\gamma_1 \in {}_p\pi^S_{(p^2-1)q-3}$ $(p \geqq 5,\ q = 2(p-1))$ *is essential.*

Since it is known that ${}_p\pi^S_{(p^2-1)q-3} \cong Z_p$, generated by $\alpha_1\beta_{p-1}$ [4], γ_1 must be a nonzero multiple of $\alpha_1\beta_{p-1}$. Thus Theorem A does not exhibit a new stable homotopy element; rather, it shows that the first element produced by the $V(n)$ construction is nontrivial.

COROLLARY.

$$\left.\begin{aligned} \alpha_1\beta_{p-1}\beta_s &= 0, & & s \geqq 3, \\ \alpha_1\beta_1\beta_k &= 0, & & k \not\equiv -2 \mod p,\ k \geqq p, \\ \alpha_1\beta_2\beta_{k-1} &= 0, & & k \not\equiv -2 \mod p,\ k \geqq p+1, \end{aligned}\right\} p \geqq 5.$$

This follows from Theorem A and Proposition 5.9 of [7].

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[1] The element $\gamma_1 \in {}_p\pi^S_{(p^2-1)q-3}$ should not be confused with the ephemeral element $\gamma \in {}_p\pi^S_{p^2q-2}$ whose nonexistence was proved by Toda [5].

THEOREM B. *The element* $\gamma_t \in {}_p\pi^S_{(tp^2+(t-1)p+(t-2))q-3}$ *is essential for* $p \geqq 7, t \leqq p-1$.

This theorem may well be true for $p = 5$. Furthermore, it is likely that all the elements $\gamma_t, t \geqq 1$, are essential. (Compare [**10**], [**12**]).

The decomposability relation $\gamma_1 = c\alpha_1\beta_{p-1}$ is a coincidence, it turns out.

COROLLARY C. *For* $2 \leqq t \leqq p-1$, γ_t *cannot be written as a linear combination of products* $\alpha_r\beta_s$.

PROOF. Toda has shown that $\alpha_r\beta_s = 0$ if $r \geqq 2$ [**7**, Theorem 5.3]. But it is easy to check that $\alpha_1\beta_s$ cannot lie in the same stem as any γ_t $(t \leqq p-1)$ unless $s = p-1$, $t = 1$. This concludes the proof.

We suspect that the γ_t are actually indecomposable for $1 < t < p$. But since even γ_2 lies beyond the range of current stable-stem computations, we have not been able to check this conjecture.

Our methods can be extended to study still higher families of elements modeled on the α's, β's, and γ's. Details (and proofs of the above theorems) will appear in a forthcoming paper by the author and Emery Thomas. I am grateful to Thomas for initiating me into the mysteries of higher order operations, to A. Liulevicius and L. Smith for helpful suggestions, and to the NSF for their support.

We detect the γ_t, following the game plan of [**10**] and [**12**], using third-order operations in the cohomology theory given by the Brown-Peterson spectrum for the prime p [**11**]. This theory, BP^*, resembles ordinary cohomology (H^*) more closely than K-theory, for example, yet has a much richer structure than H^*. It carries just as much information at the given prime as complex cobordism, MU^*, but its algebra of primary operations is much simpler. Besides, it is *sparse*, in the sense that the "Steenrod algebra" $BP^*\cdot(BP)$ is zero unless $* \equiv 0 \bmod q$, where $q = 2(p-1)$.

This sparseness comes in very handy in the detection of the γ_t. For it is not clear how to construct Postnikov systems or higher order operations in a generalized multiplicative cohomology theory, since there is no Serre spectral sequence and hence no notion of transgression. An added difficulty is that while a relation among primary operations gives rise to a secondary operation in any theory [**12**], an algebraic relation among relations may not yield any tertiary operation in the case of H^* [**3**, p. 127]. By generalizing the procedure of §1 of [**12**], however, the author and Thomas can show that any algebraic relation among relations in a sparse cohomology theory gives rise to a tertiary operation.

In our case, the relations among primary operations that matter are

$$w_p : (-p + v_1r_1)r_p + (r_{p-1} - v_1r_p)r_1 = 0 \quad (\text{see [**12**]})$$

and

$$w_{p+2}: r_2 r_p + [-r_{0,1} - \tfrac{1}{2}(p+1) r_{p+1} + \tfrac{1}{2} v_1 r_{1,1}] r_1 = 0.$$

The relation

$$x_{p+1}: r_1 w_p - 2 v_1 w_{p+2} = 0$$

gives rise to the tertiary operation Ξ we need.

Evaluating Ξ requires the use of Peterson-Stein-type formulas as in [**12**], only twice, to reduce the problem to algebra involving primary operations. The indeterminacy of Ξ may also be expressed in terms of primary operations. Both of these steps depend on sparseness.

Now let λ_0 and λ_1 be the canonical BP^*-generators (in dimensions 0 and $(tp^2 + (t-1)p + (t-2))q - 2$ respectively) of $BP^*(C_{\gamma_t})$, the BP-cohomology of the mapping cone of γ_t. Then

THEOREM D. *$\Xi(\lambda_0)$ is a nonzero multiple of λ_1* mod *indeterminacy.*

Theorems A and B follow.

NOTE ADDED IN PROOF. In September 1972, S. Oka and Toda informed us that they could show $\gamma_1 = 0$, contradicting our Theorem A. As of this writing, they have not furnished a complete proof, and the conflict is still unresolved.

REFERENCES

1. J. F. Adams, *On the groups $J(X)$*. IV, Topology **5** (1966), 21–71; correction, ibid. **7** (1968), 331. MR **33** #6628; MR **37** #5874.

2. E. H. Brown, Jr. and F. P. Peterson, *A spectrum whose Z_p-cohomology is the algebra of reduced pth powers*, Topology **5** (1966), 149–154. MR **33** #719.

3. C. R. F. Maunder, *Cohomology operations of the Nth kind*, Proc. London Math. Soc. (3) **13** (1963), 125–154. MR **35** #2279.

4. H. Toda, *p-primary components of homotopy groups*. IV. *Compositions and toric constructions*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **32** (1959), 297–332. MR **22** #1906.

5. ———, *An important relation in homotopy groups of spheres*, Proc. Japan Acad. **43** (1967), 839–842. MR **37** #5872.

6. ———, *On spectra realizing exterior parts of the Steenrod algebra*, Topology **10** (1971), 53–65. MR **42** #6814.

7. ———, *Algebra of stable homotopy of Z_p-spaces and applications*, Kyoto J. Math. **11** (1971), 197–251.

8. L. Smith, *On realizing complex bordism modules*. I. *Applications to the stable homotopy of spheres*, Amer. J. Math. **92** (1970), 793–856. MR **43** #1186a.

9. ———, Private communication.

10. L. Smith and R. Zahler, *Detecting stable homotopy classes by primary BP cohomology operations*, Math. Z. **129** (1972), 137–156.

11. R. Zahler, *The Adams-Novikov spectral sequence for the spheres*, Ann. of Math. (2) **96** (1972), 408–504.

12. ———, *Detecting stable homotopy with secondary cobordism operations*, I (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, Douglass College, New Brunswick, New Jersey 08903