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COHOMOLOGY OF BRAID SPACES

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1. **Introduction and results.** Let M be a manifold; define $F(M, k)$ to be the subspace $\{\langle x_1, \ldots, x_k\rangle | x_i \in M,\ x_i \neq x_j \text{ if } i \neq j\}$ of M^k. There is a proper right action of Σ_k, the symmetric group on k-letters, on $F(M, k)$ given by

$$\sigma \cdot \langle x_1, \ldots, x_k\rangle = \langle x_{\sigma(1)}, \ldots, x_{\sigma(k)}\rangle, \qquad \sigma \in \Sigma_k.$$

Let $B(M, k)$ denote the orbit space $F(M, k)/\Sigma_k$. The object of this paper is to outline the calculation of

$$H^*(\mathrm{Hom}_{\Sigma_p}(C_*F(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(q))), \qquad n \geqq 2, p \text{ prime},$$

where $C_*F(\boldsymbol{R}^n, p)$ denotes the singular chains of $F(\boldsymbol{R}^n, p)$, and $\boldsymbol{Z}_p(q)$ denotes the Σ_p-module $\boldsymbol{Z}_p$ with Σ_p-action $\sigma \cdot x = (-1)^{qs(\sigma)}x$ for $x \in \boldsymbol{Z}_p$ and $\sigma \in \Sigma_p$ ($(-1)^{s(\sigma)}$ is the sign of σ). Since the Σ_p-action on $F(\boldsymbol{R}^n, p)$ is proper, we may identify $H^*(\mathrm{Hom}_{\Sigma_p}(C_*F(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(2q)))$ with $H^*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p)$ [**5**]. By abuse of notation we denote $H^*(\mathrm{Hom}_{\Sigma_p}(C_*F(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(q)))$ by $H^*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(q))$.

The interest in $H^*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(q))$ arises from the work of Peter May [**6**], [**7**] which implies that each class in $H_*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(q))$ determines a homology operation on all classes of degree q in the mod p homology of any n-fold loop space.

For our calculations, we rely heavily on the map of fibrations

$$\begin{array}{ccc} \Sigma_p & \longrightarrow & \Sigma_p \\ \downarrow & & \downarrow \\ F(\boldsymbol{R}^n, p) & \xrightarrow{\hat{f}} & F(\boldsymbol{R}^\infty, p) \\ \downarrow & & \downarrow \\ B(\boldsymbol{R}^n, p) & \xrightarrow{f} & B(\boldsymbol{R}^\infty, p) \end{array}$$

where $F(\boldsymbol{R}^\infty, p) = \varinjlim F(\boldsymbol{R}^n, p)$ and $B(\boldsymbol{R}^\infty, p) = F(\boldsymbol{R}^\infty, p)/\Sigma_p$. Here f and $\hat{f}$ are the evident inclusions. Since $F(\boldsymbol{R}^\infty, p)$ is contractible with free Σ_p-action, $B(\boldsymbol{R}^\infty, p)$ is a $K(\Sigma_p, 1)$. Obviously

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$$f^*: H^*(\Sigma_p; \boldsymbol{Z}_p(q)) \to H^*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(q))$$

is defined; the structure of $H^*(\Sigma_p; \boldsymbol{Z}_p(q))$ is well known [**6**].

To facilitate the statement of our result, we recall that the product $A \,\pi\, B$ in the category of connected $\boldsymbol{Z}_p$-algebras is defined by $(A \,\pi\, B)_0 = \boldsymbol{Z}_p$ and $(A \,\pi\, B)_q = A_q \times B_q$ for $q > 0$, with product specified by $A_q \cdot B_r = 0$ for $q > 0$ and $r > 0$ and the requirement that the projections be morphisms of algebras.

THEOREM I. *For p an odd prime and* $n \geqq 2$,

$$H^*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(2q)) = A_n \,\pi\, \operatorname{Im} f^*$$

as a connected $\boldsymbol{Z}_p$*-algebra. Here* $\operatorname{Im} F^* \approx H^*(\Sigma_p; \boldsymbol{Z}_p(2q))/\operatorname{Ker} f^*$ *where* $\operatorname{Ker} f^*$ *is the ideal consisting of all elements of degree greater than* $(n-1)(p-1)$ *and*

$$A_n = E[\alpha] \quad \text{if } n \text{ is even,}$$
$$= \boldsymbol{Z}_p \quad \text{if } n \text{ is odd,}$$

where α *is an element of degree* $n-1$ *and* $E[\alpha]$ *denotes the exterior algebra on* α*. Furthermore,* α *restricts to the dual of a spherical class in the homology of* $F(\boldsymbol{R}^n, p)$*, and the Steenrod operations on* α *are trivial.*

THEOREM II. *For p an odd prime and* $n \geqq 2$,

$$H^*(B(\boldsymbol{R}^n, p); \boldsymbol{Z}_p(2q+1)) = M_n \oplus \operatorname{Im} f^*$$

as a $\boldsymbol{Z}_p$*-module. Here* $\operatorname{Im} f^* \approx H^*(\Sigma_p; \boldsymbol{Z}_p(2q+1))/\operatorname{Ker} f^*$ *where* $\operatorname{Ker} f^*$ *is the sub* $\boldsymbol{Z}_p$*-module consisting of all elements of degree greater than* $(n-1)(p-1)$ *and*

$$M_n = 0 \quad \text{if } n \text{ is even,}$$
$$= \boldsymbol{Z}_p \cdot \lambda \quad \text{if } n \text{ is odd,}$$

where λ *is an element of degree* $(n-1)((p-1)/2)$ *and* $\boldsymbol{Z}_p \cdot \lambda$ *denotes the* $\boldsymbol{Z}_p$*-vector space with basis* λ*. Furthermore,* λ *restricts to the* mod *p reduction of an integral class in* $H^*F(\boldsymbol{R}^n, p)$.

For the case $p = 2$, we have

PROPOSITION III. $B(\boldsymbol{R}^n, 2)$ *has the homotopy type of* $\boldsymbol{R}P^{n-1}$.

Finally, we remark that the spaces $B(M, j)$ were studied by Fadell and Neuwirth [3]. By specializing M to $\boldsymbol{R}^2$, Fox and Neuwirth [**4**] showed that $B(\boldsymbol{R}^2, j)$ is a $K(B_j, 1)$ where B_j is the braid group defined by Artin [**1**]. We briefly recall Fox and Neuwirth's method. They define an equivariant cell decomposition for $F(\mathbf{R}^2, j)$ and consider the induced cell decomposition for $B(\mathbf{R}^2, j)$. Here each oriented $(2j-1)$-cell represents a generator for $\pi_1 B(\mathbf{R}^2, j)$. "Small" loops about each $(2j-2)$-cell determine a complete set of relations for the generators. A calculation reveals that $\pi_1 B(\mathbf{R}^2, j) = B_j$. Since $\pi_i B(\mathbf{R}^2, j) = 0$ for $i > 1$, $B(\mathbf{R}^2, j)$ is a $K(B_j, 1)$.

Details of the calculations of $H^*[B(\mathbf{R}^n, p); \mathbf{Z}_p(q)]$ will appear elsewhere, along with a complete theory of homology operations on n-fold loop spaces.

2. **Outline of calculations.** Since the action of Σ_k on $F(\mathbf{R}^n, k)$ is free, we can apply the spectral sequence for a covering [**2**] to the covering projection $F(\mathbf{R}^n, k) \to B(\mathbf{R}^n, k)$. To calculate E_2 of this spectral sequence, we must explicitly determine the structure of $H^*F(\mathbf{R}^n, k)$ as a Σ_k-module (its additive structure is determined by use of the Serre spectral sequence and the work of Fadell and Neuwirth [**3**]). To this end, we first construct certain representative cycles α_{ij}, $1 \leqq j \leqq i \leqq k-1$, and then calculate geometrically the Σ_k-action on the resulting classes $\{\alpha_{ij}\}_* \in H_*(F(\mathbf{R}^n, k); \mathbf{Z}_p)$. Since $H^* = (H_*)^*$ here, we dualize and read off the action of Σ_k on the indecomposable elements α_{ij}^*. A calculation of the algebra structure of $H^*F(\mathbf{R}^n, k)$ in terms of the α_{ij}^* finishes the determination $H^*F(\mathbf{R}^n, k)$ as a Σ_k-module.

Next, instead of attempting to evaluate E_2^{**} directly, where $\{E_r\}$ is the spectral sequence which converges from

$$E_2^{**} = H^*(\Sigma_p; H^*(F(\mathbf{R}^n, p); \mathbf{Z}_p(q))) \quad \text{to} \quad H^*(B(\mathbf{R}^n, p); \mathbf{Z}_p(q)),$$

we study $E_2'^{**}$ where $\{E_r'\}$ is the spectral sequence obtained by replacing Σ_p with π_p, the cyclic group of order p. By careful algebraic analysis of E_2' and application of the restriction map $i(\Sigma_p : \pi_p) : \pi_p \to \Sigma_p$, we prove the following theorem:

THEOREM IV [VANISHING THEOREM]. *In the spectral sequences* $\{E_r\}$ *and* $\{E_r'\}$, $E_2^{s,t} = E_2'^{s,t} = 0$, *for* $s > 0$ *and* $0 < t < (n-1)(p-1)$.

From the fact that $B(\mathbf{R}^n, p)$ is a pn-dimensional manifold, the vanishing theorem, and Swan's results [**8**] applied to the p-period of Σ_p, we deduce most of the nontrivial differentials and almost all of E_2^{**}. To complete the additive determination of E_2^{**}, we calculate $E_2^{0,*}$, the points in $H^*(F(\mathbf{R}^n, p); \mathbf{Z}_p(q))$ fixed under the action of Σ_p. All remaining differentials and the determination of E_∞ follow directly.

We finish by indicating how the algebra structure and Steenrod operations are calculated in $H^*(B(\mathbf{R}^n, p); \mathbf{Z}_p(2q))$. Let T denote an automorphism of $\mathbf{R}^n$ given by reflection through a fixed coordinate. The map $\hat{T}: F(\mathbf{R}^n, p) \to F(\mathbf{R}^n, p)$ given by $\hat{T}\langle x_1, \ldots, x_p\rangle = \langle Tx_1, \ldots, Tx_p\rangle$ induces the obvious π_2-actions on $F(\mathbf{R}^n, p)$ and $B(\mathbf{R}^n, p)$; we note that the covering projection, $\pi : F(\mathbf{R}^n, p) \to B(\mathbf{R}^n, p)$ is π_2-equivariant. The class α of Theorem I is uniquely specified by the conditions $\hat{T}\alpha = -\alpha$ and $\pi^*\alpha \neq 0$. From the fact that $\hat{T}$ fixes all classes in $\operatorname{Im} f^*$, the remaining properties of α follow trivially.

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