BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

AN ALGEBRAIC PROOF OF AN ANALYTIC RESULT OF SHUCK'S

BY L. G. ROBERTS

Communicated by Hyman Bass, November 29, 1972

In [1] the authors pose the following problem: Let $F(X_1, \ldots, X_r)$ be a polynomial in r variables with integer coefficients. Let p be a prime. For each positive integer n let c_n be the number of solutions modulo p^n of the congruence $F(x_1, \ldots, x_r) \equiv 0 \bmod (p^n)$. Form the Poincaré series of F

$$P_F(t) = \sum_{n=1}^{\infty} c_n t^n.$$

Is $P_F(t)$ a rational function in t?

John Shuck [3] generalized this problem and then answered it affirmatively in the "nonsingular" case. His approach was to develop a calculus on manifolds over p-adic fields (including a theory of integration on submanifolds, Fubini's theorem, and change of variables), and then to utilize analytic techniques for counting.

In the following we present another reformulation of the problem, this time in an algebraic geometric setting, and give a purely algebraic and quite elementary proof in the smooth case.

My thanks here are to M. J. Greenberg who insisted that this should be written down.

1. **A reformulation.** We consider the following situation: S is a scheme (that is, a prescheme in the sense of EGA), X is an S-scheme, I a quasi-coherent ideal of O_S. Let S_n be the subscheme of S defined by the ideal I^n. Then we have the cartesian diagram

$$\begin{array}{ccc} X & \xleftarrow{\;j_n\;} & X_n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_n} \\ S & \xleftarrow[\;i_n\;]{} & S_n \end{array}$$

Let $\Gamma_n = \mathrm{Hom}_{S_n}(S_n, X_n) = \mathrm{Hom}_S(S_n, X)$. We pose the following problem: If $c_n = \mathrm{card}(\Gamma_n)$ is finite for all n greater than some N, set

AMS (MOS) subject classifications (1970). Primary 14G05, 14F10.
Key words and phrases. Sections over local schemes, Poincaré polynomial, relative differentials, regular ideal, smooth relative schemes.

$$P_{X/S,I}(t) = \sum_{n \geq N} c_n t^n.$$

When is $P_{X/S,I}(t)$ rational?

REMARK. We do not take this formulation seriously; we put it down only to give a convenient way of discussing various hypotheses that one might impose. In fact even with X/S smooth, S affine, we must still impose very stringent conditions on I to obtain the rationality.

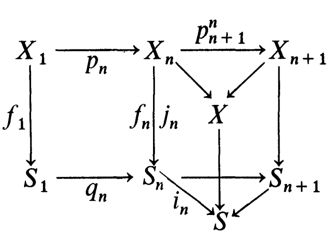
2. **The main result.** We will prove the following

THEOREM. *Let X/S be smooth, where S is the spectrum of a noetherian ring A, I an ideal of A generated by a regular element x of A. Further assume that A/I is finite and that the sets Γ_n are finite for all n (this is automatic if X is affine, or if S_n is local for all n and X is quasi-compact). Then $P_{X/S,I}(t)$ is rational.*

REMARK. If the assumptions are as in the theorem, but with I generated by an A-regular sequence x_1, x_2, then $P_{X/S,I}(t)$ is not in general rational.

PROOF. We first note that the assumptions imply immediately that A/I^n is finite for all n. The parenthetical remark then follows since in the affine case X is then $\mathrm{Spec}(A[T_1, \ldots, T_n]/(f_1, \ldots, f_r))$, and in the local case every S-morphism of S_n into X factors through some U_i where $U_1, \ldots, U_k$ is a finite affine cover of X.

Now consider the diagram which we shall use consistently:



First note that there is a natural map $\alpha_n^{n+1} : \Gamma_{n+1} \to \Gamma_n$, induced essentially by composition with $q^n_{n+1} : S_n \to S_{n+1}$. Let $\beta_{n+1} : \Gamma_{n+1} \to \Gamma_1$ designate the composition of the corresponding α_i^{i+1}.

The idea is to study the maps α_n^{n+1}. Let J_n be the ideal of $\mathrm{Spec}(A_{n+1}) = \mathrm{Spec}(A/I^{n+1})$ which defines the closed subscheme S_n. Then $J_n = (I^n/I^{n+1})$ and has square zero. Since X_{n+1}/S_{n+1} is smooth it follows immediately from the definition of smoothness [EGA IV, 17.3.1] that α_n^{n+1} is onto.

Now fix an element u of Γ_n, and consider the corresponding morphism $u_0 : S_n \to X_{n+1}$, obtained by composition with p^n_{n+1}. Let G be the O_{S_n}-module $\mathrm{Hom}_{O_{S_n}}(u_0^* \Omega^1_{X_{n+1}/S_{n+1}}, J_n)$ (J_n, having square zero, is, in a natural way, an O_{S_n}-module). Finally (following [EGA IV, 16.5.14]) introduce the sheaf of sets P on S_n defined as follows: for every open U_0 of S_n let U be the corresponding open set in S_{n+1} (which is well defined since S_n and S_{n+1} have the same underlying topological space); let $P(U_0) =$ the set of S_{n+1} morphisms from U to X_{n+1} which agree with u_0 on U_0. Now since S is affine, and since X/S is smooth it follows immediately from [EGA IV, 16.5.17 and 16.5.18] that P is, first a G-torseur (smoothness), and secondly a trivial G-torseur (affineness), that is to say, isomorphic to G (as G-torseurs). Hence we obtain that the cardinality of the fibre $(\alpha_n^{n+1})^{-1}(u)$ is precisely that of $\mathrm{Hom}_{O_{S_n}}(u_0^*\Omega^1_{X_{n+1}/S_{n+1}}, J_n)$. Since we are assuming that all the Γ_n are finite, it follows that this cardinality is also finite.

We now study this module. With u, u_0 as above, let v_0 in Γ_1 be the image of u_0 under β_n. Then we have the

LEMMA. *There is an isomorphism*

$$\mathrm{Hom}_{O_{S_n}}(u_0^*\Omega^1_{X_{n+1}/S_{n+1}}, J_n) \simeq \mathrm{Hom}_{O_{S_1}}(v_0^*\Omega^1_{X_1/S_1}, J_1).$$

PROOF. With reference to the diagram above, it follows easily that $p_{n+1}v_0 = u_0 q_n$, hence $j_{n+1}p_{n+1}v_0 = j_{n+1}u_0 q_n$, or that $j_1 v_0 = j_{n+1}u_0 q_n$. Then we have the sequence of natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{O_{S_1}}(v_0^*\Omega^1_{X_1/S_1}, J_1) &= \mathrm{Hom}_{O_{S_1}}(v_0^* j_1^*\Omega^1_{X/S}, J_1)\\ &= \mathrm{Hom}_{O_{S_1}}(q_n^* u_0^* j_{n+1}^*\Omega^1_{X/S}, J_1)\\ &= \mathrm{Hom}_{O_{S_n}}(u_0^*\Omega^1_{X_{n+1}/S_{n+1}}, q_n^* J_1).\end{aligned}$$

Finally we make the observation that due to the assumptions on I there is a (noncanonical) isomorphism between I/I^2 and I^n/I^{n+1}. From this it follows immediately that $q_n^* J_1$ is isomorphic as an O_{S_n}-ideal to J_n, and we are finished with the lemma.

We can now complete the proof of the theorem. Let $v_1, \dots, v_k$ be the members of Γ_1. Each Γ_n splits up as a direct sum of sets $\Gamma_{n,i} = \beta_n^{-1}(v_i)$, with cardinality $c_{n,i}$ say. Let c_n be the cardinality of Γ_n, and r_i that of $\mathrm{Hom}_{O_{S_1}}(v_i^*\Omega^1_{X_1/S_1}, J_1)$. Then $c_{n+1,i} = r_i c_{n,i} = r_i^n$. The rationality is then clear, and we find that $P_{X/S,I}(t) = \sum_{i=1}^k t/(1 - r_i t)$.

3. REMARKS. (a) We indicate the nonrationality for (slightly) more general I. Say that I is generated by an A-regular sequence $x_1, \dots, x_d$. Then we know by the quasi-regularity of I that the natural map $(A/I)[T_1, \dots, T_d] \simeq \mathrm{Gr}_I(A)$ is an isomorphism of graded rings. If, in addition, $\Omega^1_{X/S}$ is free of finite rank (it is locally free in any case), one can easily compute the cardinalities of the $\mathrm{Hom}_{O_{S_n}}(u_0^*\Omega^1_{X_{n+1}/S_{n+1}}, J_n)$. One finds that if $d \geqq 2$, the Poincaré series can never be rational, the growth of the c_n being too rapid.

(b) If X were a group scheme we could by a homogeneity argument obtain that the various $\mathrm{Hom}_{O_{S_1}}(v_1^*\Omega^1_{X_1/S_1}, J_1)$'s are isomorphic to the one corresponding to the unit section.

(c) The germ of the above argument is contained in [**2**, §19, Proposition 20] and, in fact, the result given there can be utilized to give another proof of the above in the setting studied by Néron (essentially affine varieties over p-adic bases).

Bibliography

1. Z. Borevich and I. Shafarevich, *Number theory*, "Nauka", Moscow, 1964; English transl., Pure and Appl. Math., vol. 20, Academic Press, New York, 1966. MR **30** #1080; MR **33** #4001.

EGA IV A. Grothendieck, *Éléments de géométrie algébrique*. IV. *Étude locale des schemas et des morphismes de schemas*. IV. Rédigés avec la collaboration de J. Dieudonné; Publ. Math. de l'I.H.É.S., no. 32, Presses Universitaires de France, Vendôme, 1967. MR **39** #220.

2. A. Néron, *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Publ. Math. de l'I.H.É.S., no. 21, Presses Universitaires de France, Vendôme, 1964. MR **31** #3423.

3. J. Shuck, *Calculus on p-adic manifolds, with applications to number theory*, Thesis, Northeastern University, 1969.

Department of Mathematics, University of British Columbia, Vancouver, British Columbia, Canada