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BOUNDED SOLUTIONS OF WHOLE-LINE DIFFERENTIAL EQUATIONS

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Let Y be a finite-dimensional linear space with norm $| \ |$, and let $R = (-\infty, \infty)$. Let $\mathscr{A}$ be the algebra of linear functions from Y to Y with induced norm $\| \ \|$, and let A be a locally integrable function from R to $\mathscr{A}$. W. A. Coppel [**2**], [**3**, Theorem 1, p. 131] has determined necessary and sufficient conditions for

$$\text{(NH)} \qquad u'(t) = f(t) + A(t)u(t)$$

to have at least one solution u in $\mathscr{L}^\infty[R^+, Y]$ (where $R^+ = [0, \infty)$) for each f in $\mathscr{L}^\infty[R^+, Y]$. R. Conti [**1**] has solved the same problem for f in $\mathscr{L}^p[R^+, Y]$, $p \geqq 1$, and u in $\mathscr{L}^\infty[R^+, Y]$. In this note we indicate how the results of [**4**] solve both of these problems for equations on R instead of R^+. Let Φ be the fundamental solution for

$$\text{(H)} \qquad v'(t) = A(t)v(t),$$

i.e., Φ is that locally absolutely continuous function from R to $\mathscr{A}$ such that $\Phi(t) = I + \int_0^t A(s)\Phi(s)\, ds$ whenever t is in R.

THEOREM. *Statements* (i) *and* (ii) *are equivalent and statements* (iii) *and* (iv) *are equivalent.*

(i) *If f is in $\mathscr{L}^\infty[R, Y]$ there is a solution u of* (NH) *in $\mathscr{L}^\infty[R, Y]$.*

(ii) *There are three supplementary projections P_{-1}, P_0, and P_1 and a number K such that if t is in R then*

$$\int_{-\infty}^t \|\Phi(t)P_{-1}\Phi(s)^{-1}\|\, ds + \left| \int_0^t \|\Phi(t)P_0\Phi(s)^{-1}\|\, ds \right| + \int_t^\infty \|\Phi(t)P_1\Phi(s)^{-1}\|\, ds \leqq K.$$

(iii) *If f is in $\mathscr{L}^p[R, Y]$, $p > 1$, there is a solution u of* (NH) *in $\mathscr{L}^\infty[R, Y]$.*

(iv) *There are P_{-1}, P_0, P_1, and K as in* (ii) *such that if t is in R then*

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$$\int_{-\infty}^{t} \|\Phi(t)P_{-1}\Phi(s)^{-1}\|^q \, ds + \left| \int_{0}^{t} \|\Phi(t)P_0\Phi(s)^{-1}\|^q \, ds \right| + \int_{t}^{\infty} \|\Phi(t)P_1\Phi(s)^{-1}\|^q \, ds \leqq K^q,$$

where $p + q = pq$.

If, in (iii), *we take* $p = 1$, *then* (iv) *can be replaced by* (iv)* *and we still have equivalence.*

$$\text{(iv)*} \quad \begin{aligned} \|\Phi(t)P_{-1}\Phi(s)^{-1}\| &\leqq K \quad \text{if } t \geqq s, \\ \|\Phi(t)P_0\Phi(s)^{-1}\| &\leqq K \quad \text{for all } (t, s) \text{ in } R \times R, \quad \text{and} \\ \|\Phi(t)P_1\Phi(s)^{-1}\| &\leqq K \quad \text{if } t \leqq s. \end{aligned}$$

The implications (ii) → (i) and (iv) → (iii) are both obvious. Let M_0 be the subspace of Y consisting of all values at zero of bounded solutions of (H). Let M_{-1} be a subspace such that $M_{-1} \oplus M_0$ is all values at zero of solutions of (H) bounded on R^+, and let M_1 be similarly determined by $R^- = (-\infty, 0]$. Let M_∞ be determined by $Y = M_0 \oplus M_{-1} \oplus M_1 \oplus M_\infty$. Let P_0, P_{-1}, P_1, and P_∞ be the projections of Y onto M_0, M_{-1}, M_1, and M_∞ respectively. If (i) holds then [**4**, Theorem 1] tells us that $P_\infty = 0$, so (ii) can now be proved with standard techniques (see [**3**, pp. 131–134]). Similarly, if (iii) holds, the integral

$$\int_{-\infty}^{\infty} \|P_\infty \Phi(s)^{-1}\|^q \, ds$$

exists, and the linear function Λ from $\mathscr{L}^p[R, Y]$ to Y given by

$$\Lambda[f] = \int_{-\infty}^{\infty} P_\infty \Phi(s)^{-1} f(s) \, ds$$

is identically zero, so $P_\infty = 0$. Now arguments similar to those of [**1**] can be used to establish (iii) → (iv).

References

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