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INDUCTION AND STRUCTURE THEOREMS FOR GROTHENDIECK AND WITT RINGS OF ORTHOGONAL REPRESENTATIONS OF FINITE GROUPS

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ABSTRACT. The Grothendieck- and Witt- ring of orthogonal representations of a finite group is defined and studied. The main application (only indicated) is the reduction of the computation of Wall's various L-groups for a finite group π to those subgroups of π, which are a semidirect product of a cyclic group γ of odd order with a 2-group β, such that any element in β acts on γ either by the identity or by taking any element in to its inverse.

Let π be a finite group and R a Dedekind ring. An $R\pi$-lattice (M, f) or just M is defined to be a finitely generated, R-projective $R\pi$-module M together with a symmetric, π-invariant nonsingular form $f: M \times M \to R$ (cf. [**3**]). For two $R\pi$-lattices M_1 and M_2 one has their orthogonal sum $M_1 \perp M_2$ and tensor product $M_1 \otimes M_2$, thus the isomorphism classes of $R\pi$-lattices form a half-ring $Y^+(R, \pi)$, whose associated Grothendieck ring is denoted by $Y(R, \pi)$. For a subgroup $\gamma \leqq \pi$ one has in an obvious way, restriction and induction of $R\pi$-lattices, resp. $R\gamma$-lattices, and it is easily seen (cf. [3]) that this makes $Y(R,\text{-})$ into a G-functor in the sense of Green (cf. [**5**]).

As in the theory of integral group-representations, where the Grothendieck ring of isomorphism classes of $R\pi$-modules is much too large for many purposes and is thus replaced by its quotient $G_0(R, \pi)$ (in the sense of [**9**]) modulo the ideal, generated by the Euler characteristics of short exact sequences of $R\pi$-modules, we are going to define certain quotients of $Y(R, \pi)$, using a relation which was first introduced by D. Quillen in [7, §5]. At first let us remark, that for any finitely generated R-projective $R\pi$-module N, one has the associated hyperbolic module $H(N) = (N \oplus N^*, f)$ with $N^* = \mathrm{Hom}_R(N, R)$ the R-dual of N, considered as $R\pi$-module (with $(g \cdot v)(n) = v(g^{-1} \cdot n)$, $g \in \pi$, $v \in N^*$, $n \in N$) and $f(N, N) = f(N^*, N^*) = 0$, $f(n, v) = v(n)$, $n \in N$, $v \in N^*$. We now define a Quillen pair (M, N) to be an $R\pi$-lattice $M = (M, f)$ together with an

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$R\pi$-submodule $N \subseteq M$, such that N is an R-direct summand (i.e., M/N and thus also N is R-projective) and $f(N, N) = 0$. For example, $(H(N), N)$ and $(H(N), N^*)$ are Quillen pairs. If (M, N) is a Quillen pair, then $N \subseteq N^\perp = \{m \in M | f(m, N) = 0\}$ and $N^\perp/N$ is an $R\pi$-lattice again.

Let I_1, resp. I_2, $\subseteq Y(R, \pi)$ be the ideal, generated by all elements of the form $[M] - [N^\perp/N] - [H(N)]$, resp. $[M] - [N^\perp/N]$, where (M, N) is a Quillen pair, and define $GU_0(R, \pi) = Y(R, \pi)/I_1$, $GW_0(R, \pi) = Y(R, \pi)/I_2$. Because $H(N) = 0$ in $GW_0(R, \pi)$ one has $I_1 \subseteq I_2$ and thus a natural surjection $GU_0(R, \pi) \twoheadrightarrow GW_0(R, \pi)$. Moreover one checks easily, that the hyperbolic construction defines a well-defined map

$$H\colon G_0(R, \pi) \to GU_0(R, \pi),$$

whose image is precisely the kernel of $GU_0(R, \pi) \to GW_0(R, \pi)$, i.e., we have a natural exact sequence $G_0(R, \pi) \to GU_0(R, \pi) \to GW_0(R, \pi) \to 0$.

Finally the G-functor structure on $Y(R, \text{-})$ carries through to a G-functor structure on $GU_0(R, \text{-})$ and on $GW_0(R, \text{-})$ and the above sequence behaves well with respect to restriction and induction.

Now for a set $\mathfrak{H}$ of subgroups of π define $I(\mathfrak{H}, GW_0)$, resp. $I(\mathfrak{H}, GU_0)$, to be the sum of the images of $GW_0(R, \gamma)$, resp. $GU_0(R, \gamma)$, $(\gamma \in \mathfrak{H})$ in $GW_0(R, \pi)$, resp. $GU_0(R, \pi)$, with respect to the induction maps from γ to π. For a set Σ of prime numbers let $\mathfrak{H}_\Sigma(\pi)$ be the set of subgroups of π, which are p-hyperelementary (i.e. have a cyclic normal subgroup of p-power index) for some $p \in \Sigma$, especially $\mathfrak{H}_\phi(\pi) = \{\gamma \leqq \pi | \gamma \text{ cyclic}\}$. For the order $|\pi| = \prod p^{\alpha p}$ of π define $|\pi|_\Sigma = \prod_{p \in \Sigma} p^{\alpha p}$, $|\pi|_{\Sigma'} = \prod_{p \notin \Sigma} p^{\alpha p}$ (thus $|\pi| = |\pi|_\Sigma \cdot |\pi|_{\Sigma'}$). Then we have

THEOREM 1. *$n \cdot 1_{GW_0(R,\pi)} \in I(\mathfrak{H}_\Sigma(\pi), GW_0)$ with $n = |\pi|_{\Sigma'}$ for $|\pi|_{\Sigma'}$ odd and $n = 4 \cdot |\pi|_{\Sigma'}$ in any case.*

THEOREM 2. *$n \cdot 1_{GU_0(R,\pi)} \in I(\mathfrak{H}_\Sigma(\pi), GU_0)$ with*

$$\begin{aligned} n &= |\pi|^2_{\Sigma'} && \text{for } R \text{ semilocal, } |\pi|_{\Sigma'} \text{ odd,} \\ &= 4 \cdot |\pi|^2_{\Sigma'} && \text{for } R \text{ semilocal,} \\ &= |\pi|^3_{\Sigma'} && \text{for } |\pi|_{\Sigma'} \text{ odd,} \\ &= 4 \cdot |\pi|^3_{\Sigma'} && \text{in any case.} \end{aligned}$$

Since $GU_0(R, \pi)$ acts naturally as a Frobenius-functor on most (if not all) of the various L-groups, associated with a finite group π (cf. [**11**]), one thus can reduce the study of these L-groups to the case of p-hyperelementary groups. Actually for any such L-functor—let it be called just L—one has

COROLLARY 1. *The various restriction maps define an isomorphism of*

$L_\Sigma(\pi) =_{\mathrm{Df}} Z[1/p|p \notin \Sigma] \otimes L(\pi)$ onto the subgroup of $\prod_{\gamma\in\mathfrak{H}_\Sigma(\pi)} L_\Sigma(\gamma)$, consisting of those tuples $(x_\gamma)_{\gamma\in\mathfrak{H}_\Sigma(\pi)}$ with $x_\gamma \in L_\Sigma(\gamma)$, $x_{\gamma|\delta} = x_\delta$ for any $\delta \leqq \gamma$ and $x_\gamma^g = x_{g\gamma g^{-1}}$ for any $g \in \pi$ ($x_{\gamma|\delta}$ the restriction from γ to δ, $x \to x^g$ the natural isomorphism from $L_\Sigma(\gamma)$ onto $L_\Sigma(g\gamma g^{-1})$, associated with g).

PROOF. This follows the same way from Theorem 2 as R. Brauer's characterization of generalized characters among class functions by their restriction to elementary subgroups from his induction theorem. Thus it can also be considered as a special case of [**1**, §8, Appendix].

Using the above exact sequence and Swan's induction theorems for $G_0(R, \pi)$ (cf. [**10**]), Theorem 2 follows by a well-known trick, due to Swan (cf. the proof of Proposition 1 in [**10**, pp. 558–559]), from Theorem 1. Theorem 1 itself follows to some extent from the following result on the structure of $GW_0(R, \pi)$:

THEOREM 3. *For any ring A write A' for $Z[\frac{1}{2}] \otimes A$. Then*

(i) $GW_0(\pi)' =_{\mathrm{Df}} GW_0(\boldsymbol{Z}, \pi)' \cong G_0(\boldsymbol{R}, \pi)'$.

(ii) $GW_0(R, \pi)' \cong GW_0(\pi)' \otimes_{\boldsymbol{Z}} W(R)'$ *(with $W(R) = GW_0(R, \varepsilon)$—$\varepsilon$ the trivial group—the Witt ring of R in the sense of Knebusch* [**6**]).

(iii) *The torsion subgroup of $GW_0(\pi)$ is annihilated by* 4.

Indeed, using the general theory of the Burnside ring (cf. [**1**, especially §8, Theorem 8.2]), Theorem 3 implies Theorem 1 with $n = 4 \cdot |\pi|_{\Sigma'}$. Theorem 3 itself follows from results of A. Fröhlich (cf. [**4**]) on $GW_0(R, \pi)$ for R a field of characteristic 0 and from

PROPOSITION 1. *If K is the quotient field of R, then the natural map $GW_0(R, \pi) \to GW_0(K, \pi)$ is injective. Furthermore, if R has no formally real residue class field, then $GW_0(R, \pi)' \to GW_0(K, \pi)'$ is an isomorphism.*

PROOF. Straightforward generalization of the argument for Satz 11.1.1 in [**6**]. Another way to prove Theorem 3 is to combine Proposition 1 with

PROPOSITION 2. *Let L be a finite Galois extension of a field K with Galois group $\mathfrak{G}$, such that any ordering of K can be extended to L. Then we have $GW_0(K, \pi)' \cong (GW_0(L, \pi)')^{\mathfrak{G}}$ for the natural action of $\mathfrak{G}$ on $GW_0(L, \pi)'$.*

This follows from [**1**, Appendix B, Theorem 3.2], using Scharlau's induction technique for Witt rings (cf. [**8**] and also [**1**, Appendix A]).

PROPOSITION 3. *Let us call a formally real field K a real splitting field for the group π, if for any irreducible $K\pi$-module N and any formally real extension L of K the module $L \otimes_K N$ is an irreducible $L\pi$-module. Then*

(i) $G_0(K, \pi) \cong G_0(\boldsymbol{R}, \pi)$,

(ii) $GW_0(K, \pi)' \cong G_0(K, \pi)' \otimes_{\boldsymbol{Z}} W(K)'$.

Moreover if $n = \exp(\pi)$, ξ a primitive nth root of unity and K any formally *real field, then* $K(\xi + \xi^{-1})$ *is a real splitting field of* π.

Propositions 1, 2 and 3 now imply the most important part of Theorem 3

$$\begin{aligned} GW_0(\pi)' &= GW_0(\boldsymbol{Z}, \pi)' \\ &\cong GW_0(\boldsymbol{Q}, \pi)' \cong (GW_0(\boldsymbol{Q}(\xi + \xi^{-1}), \pi)')^{\mathfrak{G}} \\ &\cong (G_0(\boldsymbol{Q}(\xi + \xi^{-1}), \pi)' \otimes_Z W(\boldsymbol{Q}(\xi + \xi^{-1}))')^{\mathfrak{G}} \\ &\cong (G_0(\boldsymbol{R}, \pi)' \otimes_Z W(\boldsymbol{Q}(\xi + \xi^{-1}))')^{\mathfrak{G}} \\ &\cong G_0(\boldsymbol{R}, \pi)' \otimes_Z (W(\boldsymbol{Q}(\xi + \xi^{-1}))')^{\mathfrak{G}} \\ &\cong G_0(\boldsymbol{R}, \pi)' \otimes_Z W(\boldsymbol{Q})' \cong G_0(\boldsymbol{R}, \pi)', \end{aligned}$$

especially all torsion in $GW_0(\pi)$ is 2-torsion. The other parts of Theorem 3 need some more care. (But for R a field K one has of course

$$\begin{aligned} WG_0(K, \pi)' &\cong (WG_0(K(\xi + \xi^{-1}), \pi)')^{\mathfrak{G}} \\ &\cong (G_0(K(\xi + \xi^{-1}), \pi)' \otimes W(K(\xi + \xi^{-1}))')^{\mathfrak{G}} \\ &\cong (G_0(\boldsymbol{R}, \pi)' \otimes W(K(\xi + \xi^{-1}))')^{\mathfrak{G}} \\ &\cong (GW_0(\pi)' \otimes W(K(\xi + \xi^{-1}))')^{\mathfrak{G}} \\ &\cong GW_0(\pi)' \otimes (W(K(\xi + \xi^{-1}))')^{\mathfrak{G}} \cong GW_0(\pi)' \otimes W(K)'; \end{aligned}$$

thus the same holds as well for Dedekind rings with no formally real residue class field. It also shows that for any field K all torsion in $GW_0(K, \pi)$ is 2-torsion, which was conjectured by A. Fröhlich in [**4**].)

To get rid of the factor 4 in Theorem 1 for $|\pi|_{\Sigma'}$ odd, one has to use multiplicative induction theory as developed, for instance, in [**2**]. Reducing trivially to the case $\Sigma = \{2\}$ and using this technique, it is enough to prove the corresponding statement for groups of rather simple types: elementary abelian groups of order p^2 (p odd), nonabelian groups of order $p \cdot q$ (p, q odd primes) and semidirect products of cyclic groups of order p with elementary abelian 2-groups, on which the cyclic group of order p acts nontrivially and irreducibly. But in all these cases the torsion part of $GW_0(\pi)$ is easily shown to be nilpotent and thus one can use the fact that, in case all torsion elements in $GW_0(R, \pi)$ are nilpotent, the wanted result follows directly from Theorem 3 by AGN-methods and Burnside ring theory (cf. [**1**, especially §8, Theorem 8.2]). Actually I conjecture that for any group π all torsion elements in $GW_0(R, \pi)$ are nilpotent. This would allow us to avoid multiplicative induction techniques in this case completely; on the other hand, our induction theorem reduces this question to the case of 2-hyperelementary groups. I can prove the conjecture for a great number of special classes of groups, but right now it seems to me, that a proof in the general case might be as complicated and even more involved than the multiplicative induction techniques I am using now.

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