BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

GROTHENDIECK AND WHITEHEAD GROUPS OF TORSION FREE ABELIAN GROUPS

BY DAVID M. ARNOLD

Communicated by Joseph Rotman, December 19, 1972

Let $\mathscr{A}$ denote the category of torsion free abelian groups of finite rank and let $K_0(\mathscr{A})$ and $K^0(\mathscr{A})$ be the Grothendieck groups of $\mathscr{A}$ (modulo split exact sequences and exact sequences, respectively).

J. Rotman [5] determined the group and ring structure of $K^0(\mathscr{A})$; in particular, $K^0(\mathscr{A})$ is a free abelian group of uncountable rank. There is a canonical epimorphism $\pi_0 : K_0(\mathscr{A}) \to K^0(\mathscr{A})$ so $K_0(\mathscr{A})$ has a free summand.

Let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ consisting of groups with constant p-rank (i.e., there is an integer n such that the Z/pZ-dimension of A/pA is equal to n for all primes p). Define $K_0(\mathscr{C})$ to be the Grothendieck group of $\mathscr{C}$ (modulo split exact sequences) and let $\tilde{K}_0(\mathscr{C})$ be the kernel of the rank homomorphism from $K_0(\mathscr{C})$ to Z, the ring of integers.

PROPOSITION 1. *$\tilde{K}_0(\mathscr{C})$ is isomorphic to the kernel of π_0.*

The category $\mathscr{C}'$ is defined by letting the objects of $\mathscr{C}'$ be the objects of $\mathscr{C}$, and with morphism sets $Q \otimes_Z \mathrm{Hom}_Z(A, B)$ for groups A and B in $\mathscr{C}'$. There is a canonical epimorphism $\sigma_0 : K_0(\mathscr{C}) \to K_0(\mathscr{C}')$. Moreover, $K_0(\mathscr{C}')$ is a free abelian group (of uncountable rank) since $\mathscr{C}'$ has a Krull-Schmidt theorem (e.g., see Walker [6]).

If R is a ring with identity, then $K_0(R)$ is defined to be $K_0(\mathscr{P}_R) = K^0(\mathscr{P}_R)$, $\mathscr{P}_R$ the category of finitely generated projective R-modules.

A corollary to the next theorem is: The torsion subgroup of $K_0(\mathscr{A})$ is nonzero (for $K_0(R) \simeq Z \oplus I(R)$, where $I(R)$ is the ideal class group of R).

THEOREM 2. *Suppose that R is a Dedekind domain such that R^+, the additive group of R, is a reduced torsion free abelian group of finite rank. Then $K_0(R)$ is isomorphic to a subgroup of $K_0(\mathscr{A})$.*

Let $K_1(\mathscr{A})$ be the Whitehead group of $\mathscr{A}$ (as defined by Bass [3]). Since $\mathscr{C}$ is a cofinal subcategory of $\mathscr{A}$, we have

COROLLARY 3. $K_1(\mathscr{C}) \simeq K_1(\mathscr{A})$.

AMS (MOS) subject classifications (1970). Primary 20K15, 18F25; Secondary 13F05, 13D15.

Key words and phrases. Torsion free abelian groups of finite rank, Grothendieck groups, Whitehead groups.

Define $\mathscr{P}(\mathscr{C})$ to be the collection of groups in $\mathscr{C}$ with constant p-rank 1 and with product $A * B = (A \otimes_Z B)/d(A \otimes_Z B)$, where $d(A \otimes_Z B)$ is the divisible subgroup of $A \otimes_Z B$. Let J be the product, over all primes p, of the p-adic integers; $U(J)$ the multiplicative group of units in J; and $AU(J)$ the group of algebraic units of J, i.e., $AU(J) = \{x \in U(J) | f(x) = 0$ for some nonzero polynomial f with integral coefficients$\}$.

THEOREM 4. (a) *There are group epimorphisms* $d_i : K_i(\mathscr{C}) \to K_i(\mathscr{P}(\mathscr{C}))$ *for* $i = 0, 1$.

(b) $K_1(\mathscr{P}(\mathscr{C})) \simeq AU(J)$.

(c) $K_1(\mathscr{P}(\mathscr{C}')) \simeq AU(J) \oplus U^+(Q)$, *where* $U^+(Q)$ *is the multiplicative group of positive rational numbers.*

Let $H : \mathscr{C} \to \mathscr{P}_J$ be the (Z-adic) completion functor. Then there is a commutative diagram

$$\begin{array}{ccc} K_1(\mathscr{C}) & \xrightarrow{K_1(H)} & K_1(J) \simeq U(J) \\ \big\downarrow d_1 & & \big\downarrow \det_1 \\ K_1(\mathscr{P}(\mathscr{C})) & \xrightarrow{K_1(H)} & K_1(\mathrm{Pic}(J)) \simeq U(J) \end{array}$$

where $\det_1$ is the usual determinant homomorphism. Furthermore, $K_1(H)$ is a monomorphism on $K_1(\mathscr{P}(\mathscr{C}))$.

The structure of $K_0(\mathscr{P}(\mathscr{C}))$ remains an open question. Some partial results are contained in [**2**]; in particular, $K_0(\mathscr{P}(\mathscr{C}))$ has a summand isomorphic to $\prod/\sum$, where $\prod$ and $\sum$ are the direct product and sum, respectively, of a countable number of copies of Z.

REFERENCES

1. D. M. Arnold and E. L. Lady, *Endomorphism rings of direct sums of torsion free abelian groups*, (to appear).

2. D. M. Arnold, *Algebraic K-theory and torsion free abelian groups of finite rank*, Symposia Mathematica, Academic Press, New York, (to appear).

3. H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR **40** #2736.

4. J. Milnor, *Introduction to algebraic K-theory*, Annals of Math. Studies 72, Princeton Univ. Press, 1971.

5. J. Rotman, *The Grothendieck group of torsion free abelian groups of finite rank*, Proc. London Math. Soc. (3) XIII (1963), 724–732. MR **27** #4857.

6. E. Walker, *Quotient categories and quasi-isomorphism of abelian groups*, Proceedings of Colloquium on Abelian Groups, Budapest, 1964, 147–162. MR **31** #2327.

DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88003