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A COMPLETE BOOLEAN ALGEBRA OF SUBSPACES WHICH IS NOT REFLEXIVE

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This note provides a negative answer to a question raised by P. R. Halmos [**2**, Problem 9]. For the convenience of the reader, the terminology necessary to understand the question is presented here. Let $\mathscr{L}$ be a lattice of subspaces of a Hilbert space $\mathscr{H}$ and let $\operatorname{Alg}\mathscr{L}$ be the algebra of all bounded operators in $\mathscr{B}(\mathscr{H})$ that leave each subspace in $\mathscr{L}$ invariant. If $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$, let $\operatorname{Lat}\mathscr{A}$ be the lattice of all subspaces of $\mathscr{H}$ that are left invariant by each operator in $\mathscr{A}$. A lattice $\mathscr{L}$ is *reflexive* if $\operatorname{Lat}\operatorname{Alg}\mathscr{L} = \mathscr{L}$. If $\mathscr{L}$ is a reflexive lattice and $\{P_i\}$ is a net of orthogonal projections such that $P_i(\mathscr{H}) \in \mathscr{L}$ for each i and $P_i \to P$ in the strong operator topology then $P(\mathscr{H}) \in \mathscr{L}$; in other words, $\mathscr{L}$ is *strongly closed.* It is true that a strongly closed lattice of subspaces is a complete lattice, but the converse is false.

A Boolean algebra of subspaces is a distributive lattice $\mathscr{L}$ such that for each M in $\mathscr{L}$ there is a unique M' in $\mathscr{L}$ such that $M \cap M' = (0)$ and $M \vee M' \equiv (M + M')^- = \mathscr{H}$. (Note that it is only required that $\mathscr{H}$ be the closure of $M + M'$.) Problem 9 of [**2**] asks: Is every complete Boolean algebra of subspaces reflexive? The answer is no, and this is shown in this paper by giving a complete Boolean algebra of subspaces which is not strongly closed. In one sense this answer seems unsatisfactory because a new question arises: Is every strongly closed Boolean algebra of subspaces reflexive? In another sense the answer is satisfying because the original question was the proper one to ask. The property of completeness is a lattice theoretic one, while the property of being strongly closed is not.

For the remaining terminology the reader is referred to [**4**] and other standard references. If $X = [0, 2\pi]$, let μ be a positive singular measure on the collection $\mathscr{A}$ of Borel subsets of X. For A in $\mathscr{A}$ define

$$\varphi_A(z) = \exp\left(-\int_A \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right), \qquad |z| < 1,$$

and put $\varphi = \varphi_X$. Each φ_A is an inner function, and φ_A is a divisor of φ_B if and only if $A \subset B$. $\mathscr{H} = H^2 \ominus \varphi H^2$ and, for each A in $\mathscr{A}$, $M_A = \varphi_A H^2 \ominus \varphi H^2$.

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$$M_A \cap M_B = M_{A \cup B}. \tag{1}$$

In fact, $\varphi_A H^2 \cap \varphi_B H^2 = \psi H^2$ where ψ is the least common multiple of φ_A and φ_B. It is easy to check that $\psi = \varphi_{A \cup B}$ and from this it follows that (1) holds. Similarly

$$M_A \vee M_B = M_{A \cap B}. \tag{2}$$

It follows from (1) and (2) that $\mathscr{L} = \{M_A : A \in \mathscr{A}\}$ is a distributive lattice; and, with $M'_A = M_{X-A}$, $\mathscr{L}$ is a Boolean algebra of subspaces of $\mathscr{H}$.

LEMMA. *$\mathscr{L} = \{M_A : A \in \mathscr{A}\}$ is a complete Boolean algebra.*

PROOF. It suffices to show that if $\mathscr{B} \subset \mathscr{A}$ then there is an A in $\mathscr{A}$ with $M_A = \bigcap \{M_B : B \in \mathscr{B}\}$. Because of (1), $\mathscr{B}$ may be assumed to be closed under finite unions. If $\beta = \sup \{\mu(B) : B \in \mathscr{B}\}$ then there is an increasing sequence $\{B_n\}$ in $\mathscr{B}$ such that $\beta = \lim_n \mu(B_n)$. If $A = \bigcup \{B_n : n \geqq 1\}$ then $\mu(A) = \beta$ and $\mu(B - A) = 0$ for every B in $\mathscr{B}$. It is claimed that $\varphi_A =$ l.c.m. $\{\varphi_B : B \in \mathscr{B}\}$. In fact, if $B \in \mathscr{B}$ then $\varphi_A = \varphi_{A-B}\varphi_B$ since $\mu(B - A) = 0$. Also, if ψ is an inner function that is a multiple of φ_B for each B in $\mathscr{B}$ then, for every integer n, $\psi = \varphi_{B_n}\psi_n$ for some inner function ψ_n. But $\varphi_{B_n}(z) \to \varphi_A(z)$ for every z so it follows that $\psi_n(z) \to \tilde{\psi}(z)$ for some inner function $\tilde{\psi}$. Hence $\psi = \varphi_A \tilde{\psi}$ and $\varphi_A =$ l.c.m. $\{\varphi_B : B \in \mathscr{B}\}$. Consequently,

$$M_A = \bigcap \{M_B : B \in \mathscr{B}\}.$$

THEOREM. *$\mathscr{L} = \{M_A : A \in \mathscr{A}\}$ is reflexive if and only if μ is a purely atomic measure.*

PROOF. If μ is purely atomic then $\mathscr{L}$ is an atomic Boolean algebra and hence is reflexive [**3**]. To prove the converse, two additional results are needed. The first of these can be found in [**5**] although the proof contains an error (which can be rectified). However, in the case under consideration (where $L^1(\mu)$ is separable) the proof is valid. (Also see [**1**].)

THEOREM A. *Let $(X, \mathscr{A}, \mu)$ be a decomposable nonatomic measure space and let $f \in L^\infty(X, \mathscr{A}, \mu)$ such that $0 \leqq f \leqq 1$. Then there is a sequence $\{A_n\}$ in $\mathscr{A}$ such that $\chi_{A_n} \to f$ in the weak-star topology of L^∞.*

THEOREM B. *For each inner function q let E_q be the orthogonal projection of H^2 onto qH^2. If $q, q_1, q_2, \ldots$ are inner functions such that $q(z) = \lim_n q_n(z)$ for $|z| < 1$ then $E_{q_n} \to E_q$ strongly.*

PROOF. If z^m is the function that assumes the value a^m at a then it is easily verified that

$$E_q(z^m) = q \sum_{k=0}^{m} \frac{1}{k!} \overline{q^{(k)}(0)}\, z^{m-k}.$$

It follows that $E_q(z^m)(a) = \lim_n E_{q_n}(z^m)(a)$ for $|a| < 1$. This gives that $E_{q_n}(z^m) \to E_q(z^m)$ weakly in H^2. Since polynomials are dense in H^2, $E_{q_n} \to E_q$ in the weak operator topology. But for projections weak convergence is equivalent to strong convergence, and the proof is complete.

Suppose μ is not purely atomic; the proof of the main theorem will be completed by showing that $\mathscr{L}$ is not strongly closed. There is a set A in $\mathscr{A}$ that contains no atoms for μ and with $\mu(A) > 0$. Let f be any Borel function such that $0 \leqq f \leqq 1$, $f(x) = 0$ for x in $X - A$, and $0 < f(x) < 1$ on a set of positive measure. According to Theorem A there is a sequence $\{A_n\}$ in $\mathscr{A}$ such that $A_n \subset A$ and $\chi_{A_n} \to f$ in the weak-star topology of $L^\infty(\mu)$. For each z, $|z| < 1$,

$$\varphi_{A_n}(z) \to \psi(z) = \exp\left(- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} f(\theta)\, d\mu(\theta) \right).$$

Theorem B implies that $E_{\varphi_{A_n}} \to E_\psi$ strongly; so $E_{\varphi_{A_n}} - E_\varphi \to E_\psi - E_\varphi$ strongly. It is straightforward to show that if P_A is the projection of $\mathscr{H}$ onto M_A, then $P_{A_n} \to P_\psi$, where P_ψ is the projection of $\mathscr{H}$ onto $\psi H^2 \ominus \varphi H^2$. Since $\psi H^2 \ominus \varphi H^2 \neq M_A$ for any A, the proof is complete.

Finally, it should be pointed out that $\mathscr{L}$ is isomorphic to the reflexive Boolean algebra Lat T, where T is multiplication by the independent variable on $L^2(X, \mu)$.

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