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THE CHARACTERS OF THE BINARY MODULAR CONGRUENCE GROUP

BY PHILIP C. KUTZKO

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1. **Introduction.** The determination of the characters of the groups $SL(2, Z/p^nZ)$ where p is an odd prime is of interest for several reasons, among them the role that the group plays in the study of elliptic modular functions [**1**], [**4**], [**5**], [**6**]. Attempts have been made to compute the values of the irreducible characters of $SL(2, Z/p^nZ)$ and complete results were obtained by Shur [**11**] in the case $n = 1$ and Praetorius [**9**] and Rohrbach [**10**] in the case $n = 2$. Since that time analytic techniques involving theta-functions [**7**] and methods used for similar problems over locally compact groups [**12**] have been applied with partial results in the first case and a classification theorem in the second.

The purpose of the note is to announce that a complete description of the irreducible representations of $SL(2, Z/p^nZ)$ as well as the computation of the characters of these representations has been obtained by the author. These results comprise the author's Ph.D. Thesis (University of Wisconsin—1972) and will be published elsewhere.

2. **Outline of results.** Write G_n for L.F.$(2, Z/p^nZ)$; i.e., let

$$G_n = SL(2, Z/p^nZ)/\{\pm\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\}.$$

For $r = 1, 2, \ldots, n$, let $K_{n,r}$ be the kernel of the homomorphism of G_n onto G_r that is the result of "reading" G_n modulo p^r. Then $K_{n,r}$ is abelian of type $(p^{n-r}, p^{n-r}, p^{n-r})$ whenever $r \geqq n/2$. Now fix n to be even and let $r = n/2$. Write $e_r(x)$ for $\exp(2\pi ix/p^r)$.

DEFINITION 1. Fix l, $0 \leqq l \leqq p^r - 1$ and write $l = p^\alpha l_1$ where $0 \leqq l_1 \leqq p^{r-\alpha} - 1$ and $p \nmid l_1$ in case $l \neq 0$ and where we take $\alpha = r$, $l_1 = 1$ in case $l = 0$. Write a typical element of $K_{n,r}$ in the form $I + p^r\left(\begin{smallmatrix}a & b\\ c & -a\end{smallmatrix}\right)$ where $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Then ψ_l is defined to be the character on $K_{n,r}$ which maps $I + p^r\left(\begin{smallmatrix}a & b\\ c & -a\end{smallmatrix}\right)$ to $e_r((lb + c)/2l_1)$.

THEOREM 1. *ψ_l may be extended to a character η_l defined on the normalizer of ψ_l (see* [**4**] *for definitions). The character χ_l induced by η_l on G_n is irreducible. χ_l does not contain $K_{n,n-1}$ in its kernel.*

Having proved Theorem 1 it is a simple matter to determine con-

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ditions on l such that the resulting set of characters χ_l form a complete set of irreducible characters on G_n which are not characters on G_{n-1} and these characters may be computed with relative ease.

Now let $n > 1$ be odd and write $r = (n+1)/2$. As in the even case write a typical element of $K_{n,r}$ in the form $I + p^r\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ and define ψ_l on $K_{n,r}$ by $\psi_l(I + p^r\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)) = e_{r-1}((lb + c)/2l_1)$. Here, $l = 0, 1, \ldots, p^{r-1} - 1$. Write T_l for the normalizer of ψ_l in G_n. Then ψ_l cannot be extended to T_l and we must modify our techniques.

THEOREM 2. *Let $p \nmid l$. Let H_l be the subgroup of elements of G_n of the form $\left(\begin{smallmatrix} t & u \\ lu & t \end{smallmatrix}\right)$ and let $T_{l,0} = H_l K_{n,r}$. Then ψ_l can be extended to a character θ_l on $T_{l,0}$. Let θ_l' be the restriction of θ_l to $T_{l,0}$ and let χ_l and χ_l' be the characters induced on G_n by θ_l and θ_l' respectively. Then $(l/p)(\chi_l - 2/p\chi_l')$ is an irreducible character on G_n which does not contain $K_{n,n-1}$ in its kernel. Here, (l/p) is the Legendre symbol.*

We note that this construction is not as unnatural as it seems since in fact $T_l = H_l K_{n,r-1}$ and since the character constructed in Theorem 2 has the same character table as the character constructed in Theorem 1 in case $p \nmid l$.

THEOREM 3. *Let $p \mid l$. Let $\bar{N}_l$ be the subset of elements of G_r of the form $\left(\begin{smallmatrix} t & u \\ l & ut \end{smallmatrix}\right)(I + p^{r-1}\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right))$. Then $\bar{N}_l$ is in fact a subgroup of G_r. Let N_l be the total inverse image of $\bar{N}_l$ in G_n. Then ψ_l can be extended to a character θ_l on N_l and the character induced by θ_l on G_n is irreducible and does not contain $K_{n,n-1}$ in its kernel.*

Again it is relatively simple to select a subset of the characters constructed in Theorems 2 and 3 which is precisely the set of irreducible characters on G_n which are not characters of G_{n-1}.

Since the characters of G_1 are known [**5**], and since for any character χ of G_n (n even or odd) there is a k such that χ may be viewed as a character on G_k but not on G_{k-1}. Theorems 1–3 determine all irreducible characters on G_n.

Finally, the methods outlined above may be utilized with minor modifications to determine the characters of $SL(2, Z/p^nZ)$ and one notes [**8**] that this suffices to determine the characters of $SL(2, Z/mZ)$ for any integer m.

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Department of Mathematics, Princeton University, Princeton, New Jersey 08540