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EXTENDING EQUIVARIANT MAPS FOR COMPACT LIE GROUP ACTIONS[1]

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ABSTRACT. Extensions of maps are studied in the category of spaces with actions of a compact Lie group G. If G acts on a finite dimensional compact metric space X with a finite number of conjugacy classes of isotropy subgroups, if $\tilde{X}$ is a closed equivariant subspace of X such that the action on $X - \tilde{X}$ is free and if $f:\tilde{X} \to Y$ is an equivariant map to a compact metric space Y with a G-action, then an equivariant neighborhood extension of f exists provided that Y is an ANR; if Y is an AR, then f can be equivariantly extended over X.

1. **Introduction.** In previous papers [**2**] and [**3**], an extension theorem for equivariant maps in the category of spaces with periodic homeomorphisms was proved. That theorem was then applied to a characterization of equivariant absolute neighborhood retracts and absolute retracts in this category. The purpose of this note is to announce results which extend some of the results of [**2**] and [**3**] from the category of $\boldsymbol{Z}_p$-actions to the category of compact Lie group actions. Detailed proofs will appear in a forthcoming paper.

The following theorem is the main result of this paper.

(1.1) THEOREM. *Let G be a compact Lie group acting on a finite dimensional compact metric space X with a finite number of conjugacy classes of isotropy subgroups; and let $\tilde{X}$ be a closed equivariant subspace of X containing all the fixed points of the elements of G different from the identity. Let G act on a compact metric space Y and let $f:\tilde{X} \to Y$ be an equivariant map. Then*:

(i) *If Y is an ANR, there exists an equivariant extension $g:U \to Y$ of f over an equivariant neighborhood U of $\tilde{X}$ in X*;

(ii) *If Y is an AR, there exists an equivariant extension $g:X \to Y$ of f over X.*

As it was pointed out in [**2**] and [**3**], the problem of equivariant extension maps is not trivial even for $\boldsymbol{Z}_2$-actions, that is, for spaces with involutions, if they are not fixed point free. The significance of this result

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lies, of course, in the fact that Y is not assumed to be an "equivariant" ANR or AR. This theorem leads, however, to a characterization of equivariant retracts; it will be the subject of a forthcoming paper [**4**].

The proof of Theorem (1.1) is based on the following idea. An action of a topological group G on a space X may be regarded as a "principal fibration with singularities", the singularities being due to the existence of fixed points. An equivariant map f determines a cross-section in the "associated singular fibration"; and the problem of extending f over the free part of the action amounts to that of extending the cross-section from the singular part to the regular part of the associated fibration. A useful tool in the construction of the extension is a linearization of a compact Lie group action due to G. D. Mostow; i.e., an equivariant embedding of the space in a Euclidean space with an orthogonal G-action.

2. **Group actions and singular fibrations.** If $p:E \to B$ is a map and $\bar{B}$, $\tilde{B}$ are complementary parts of B then the restrictions $\bar{p}:\bar{E} \to \bar{B}$, and $\tilde{p}:\tilde{E} \to \tilde{B}$, where $\bar{E} = p^{-1}\,\bar{B}$ and $\tilde{E} = p^{-1}\,\tilde{B}$, will be called complementary parts of p; we shall also say that $\bar{p}$ is the part of p over $\bar{B}$, and the partition of p into $\bar{p}$ and $\tilde{p}$ will be denoted by $p = (\bar{p}|\tilde{p})$. The part $\bar{p}$ is said to be open (resp. closed) if $\bar{B}$ is open (resp. closed).

If $p = (\bar{p}|\tilde{p})$ is a partition of $p:E \to B$ such that $\bar{p}$ is a locally trivial (numerable) fibration then p will be said to be a singular fibration with a regular part $\bar{p}$ and a singular part $\tilde{p}$.

Let Top^G be the category of left actions of a group G on topological spaces. Its objects are maps $\alpha:G \times X \to X$ satisfying the usual conditions; its subobjects are also called equivariant subspaces; and the morphisms in Top^G are also called equivariant maps. If $\bar{X}$ and $\tilde{X}$ are complementary equivariant subspaces of X with an action $\alpha:G \times X \to X$, then, just as above, we speak of complementary actions $\bar{\alpha}:G \times \bar{X} \to \bar{X}$ and $\tilde{\alpha}:G \times \tilde{X} \to \tilde{X}$ defined by α on $\bar{X}$ and $\tilde{X}$ and we write $\alpha = (\bar{\alpha}|\tilde{\alpha})$. If $\bar{\alpha}$ is free, then $\bar{\alpha}$ is called a regular part of α; and $\tilde{\alpha}$ is then the corresponding singular part. The corresponding identification maps to the orbit spaces will be denoted by $p = p^\alpha:X \to X/\alpha$, $\bar{p} = p^{\bar{\alpha}}:\bar{X} \to \bar{X}/\bar{\alpha}$ and $\tilde{p} = p^{\tilde{\alpha}}:\tilde{X} \to \tilde{X}/\tilde{\alpha}$. Provided that G is a compact Lie group and X is completely regular, $\bar{p}$ is then a principal G-fibration. For this reason, $p = (\bar{p}|\tilde{p})$ is called a singular principal G-fibration with a regular part $\bar{p}$ and a singular part $\tilde{p}$.

If $\alpha:G \times X \to X$ is a G-action and $f:B \to X/\alpha$ is a map, then f induces in a natural way a G-action $\beta = f^*\alpha:G \times Z \to Z$, where Z is the space of the fibration $f^*(p^\alpha)$ induced by f from p^α.

Let $\alpha:G \times X \to X$ and $\beta:G \times Y \to Y$ be actions of G on spaces X and Y. Then by the G-action on $X \times Y$ associated to α and β we mean the composition

$$G \times X \times Y \xrightarrow{(\text{diagonal}) \times 1_{X \times Y}} G \times G \times X \times Y$$

$$\cong G \times X \times G \times Y \xrightarrow{\alpha \times \beta} X \times Y.$$

If $\gamma : G \times X \times Y \to X \times Y$ is the action associated to α and β then the first projection $X \times Y \to X$ is an equivariant map $\gamma \to \alpha$ and thus induces a map

$$q = q^{\gamma} : (X \times Y)/\gamma \to X/\alpha.$$

The map q will be called the singular fibration associated to the singular principal fibration p^{α} and to the action β. To a regular part $\bar{\alpha}$ of α there corresponds a regular part $\bar{q}$ of q; it is then just the fibration with fibre Y associated to the principal fibration $p^{\bar{\alpha}}$, the regular part of p.

3. **Equivariant maps and cross-sections.** It was first pointed out by A. Heller [**1**] that an equivariant map $f : X \to Y$ determines a cross-section $\varphi^f : X/\alpha \to (X \times Y)/\varphi$ of q^{γ}; specifically, φ^f is the quotient map induced by the equivariant map $(1, f) : X \to X \times Y$. We shall call φ^f the cross-section associated to f. Conversely, Heller proved that if the principal fibration p^{α} is regular, then every cross-section φ of the fibration q^{γ} associated to p^{α} and β determines an equivariant map $f : X \to Y$ such that $\varphi^f = \varphi$. The following theorem is a relativization of Heller's result to the case of singular actions:

(3.1) THEOREM. *Let α and β be actions of a topological group G on spaces X and Y as in* (1.1) *and let $p = p^{\alpha} : X \to X/\alpha$ be the corresponding singular principal fibration. Let $q = q^{\gamma} : (X \times Y)/\gamma \to X/\alpha$ be the singular fibration associated to p and β. Suppose that the singular part $\tilde{\alpha} : G \times \tilde{X} \to \tilde{X}$ of α is such that $\tilde{X}$ is closed in X. Let $\tilde{p} : \tilde{X} \to \tilde{X}/\tilde{\alpha}$ be the singular part of p and let*

$$\gamma : G \times X \times Y \to X \times Y, \qquad \tilde{\gamma} : G \times \tilde{X} \times Y \to \tilde{X} \times Y$$

be the actions associated to α, β and $\tilde{\alpha}$, β, respectively. Let $f : \tilde{X} \to Y$ be an equivariant map and $\varphi^f : \tilde{X}/\tilde{\alpha} \to (\tilde{X} \times Y)/\tilde{\gamma}$ be the cross-section associated to f. Then f has an equivariant extension $g : X \to Y$ over X if and only if the cross-section φ^f has an extension to a cross-section of the fibration q.

4. **Extending cross-sections in singular fibrations.** In view of (3.1), Theorem (1.1) reduces to the following theorem:

(4.1) THEOREM. *Let α be an action of a compact Lie group G on a finite dimensional compact metric space with a finite number of conjugacy classes of isotropy subgroups. Let $p = p^{\alpha} : X \to X/\alpha$ be the corresponding singular fibration and $\tilde{p} : \tilde{X} \to \tilde{X}/\tilde{\alpha}$ be a closed singular part of p. Let β be a G-action* *on a compact metric space Y and let q and $\tilde{q}$ be the singular fibrations associated to p, β and $\tilde{p}$, β respectively.*

(i) *If Y is an ANR, then any cross-section of $\tilde{q}$ can be extended to a cross-section of q over a neighborhood of $\tilde{X}/\tilde{\alpha}$;*

(ii) *If Y is an AR, then any cross-section of $\tilde{q}$ can be extended to a cross-section of q.*

OUTLINE OF THE PROOF. Applying Mostow's linearization theorem [**5**] we can assume that X is equivariantly embedded in a Euclidean space R^n with an orthogonal G-action.

Let $\overline{R^n}$ be the maximal part of R^n on which the action is free. Let $E = \overline{R^n} \cup X, \bar{E} = E - \tilde{X}, \tilde{E} = \tilde{X}$. We replace the action α by this orthogonal action on E and continue denoting it by α with the partition $\alpha = (\bar{\alpha}|\tilde{\alpha})$ into the regular and singular parts, $\bar{E}$ and $\tilde{E}$, respectively. We also keep the notation $p = (\bar{p}|\tilde{p})$ for the corresponding orbit maps. The regular part $\bar{E}$ is an open equivariant subset of R^n and the base space $\bar{E}/\bar{\alpha}$ is an open manifold. Let $B = E/\alpha$, $\bar{B} = \bar{E}/\bar{\alpha}$, and $\tilde{B} = \tilde{E}/\tilde{\alpha}$. We would like to replace $\bar{B}$ by an infinite simplicial complex and for this purpose we prove the following lemma (compare Lemma 4.8 of [**2**]):

(4.2) LEMMA. *There exist:*

(1) *a space Z containing $\tilde{B}$ as a closed subset;*

(2) *a finite dimensional locally finite triangulation K of $Z - \tilde{B}$;*

(3) *maps of pairs*

$$\kappa:(B, \bar{B}) \to (Z, |K|) \qquad \lambda:(Z, |K|) \to (B, \bar{B})$$

each being the identity on $\tilde{B}$;

(4) *A homotopy $\lambda \circ \kappa \simeq 1_{(B,\bar{B})}$ fixing every point on $\tilde{B}$.*

Now for the proof of (4.1) it suffices to construct a cross-section of the induced singular fibration $\lambda^* q$ whose regular part is over the infinite polyhedron $|K|$. This can be done by a stepwise extension on the skeletons of K. It is interesting, however, that the finite dimensionality of K plays an essential role in the proof, since a special care is needed for simplices approaching the singular part of $\tilde{B}$.

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