BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 4, July 1973

BOOK REVIEW

Fourier Series and Integrals, by H. Dym and H. P. McKean. Probability and Mathematical Statistics No. 14, Academic Press, New York and London, 1972. $18.50.

Harmonic analysis has had more than its share of fine books. Starting with Fourier's great essay *Théorie Analytique de la Chaleur* ("the bible of the mathematical physicist," according to Arnold Sommerfeld), the list is surprisingly, almost embarrassingly, long. In addition to such basic books as Wiener's groundbreaking *The Fourier Integral and Certain of Its Applications* and Zygmund's fundamental treatise *Trigonometrical Series*, there have been volumes like Loomis' *Abstract Harmonic Analysis* and Rudin's *Fourier Analysis on Groups* which, by popularizing and developing new ideas, have exerted an important influence on research and the creation of new mathematics. More recently, we have seen the publication of Edwards' two-volume *Fourier Series* and Katznelson's exceptionally fine text *Introduction to Harmonic Analysis.* And, among the more specialized monographs, one must mention, in addition to Salem's lovely little *Algebraic Numbers and Fourier Analysis*, *Ensembles Parfaits et Séries Trigonométriques* by Kahane-Salem and Kahane's two volumes, *Some Random Series of Functions* and *Séries de Fourier Absolument Convergentes.* Lists, like comparisons, are invidious: ours could be lengthened at will.

What has been missing up to now is a text, intelligible to a fairly broad mathematical public, which explains just what Fourier analysis is all about; which makes explicit the connections with probability and number theory, elliptic functions and differential equations, electrical engineering and information theory and quantum mechanics; which puts it all together. Now Harry Dym and Henry McKean have written that book.

It is a subject for wonder that it was not written earlier. Certainly, the current appeal of harmonic analysis is very great. Recent brilliant advances in the subject (most notably, the work of Malliavin, Carleson, Varopoulos, Fefferman, and others) together with the rich amalgamation of techniques and ideas and the foudroyant elegance of the results combine to give the subject an almost irresistible attraction. Yet most of this work has dealt with the fine analysis of functions: problems of pointwise convergence, spectral synthesis, and thin sets, surely the very sort of thing the talmudic sage must have had in mind when he consigned mathematics to the "periphery of wisdom." Important as these developments are from the point of view of the researcher, their emphasis has obscured for most, if not all, mathematics students the vital connections harmonic analysis has with other branches of mathematics and science.

As a result, we have the not uncommon phenomenon of the graduate student who may know that the Fourier transform is a special case of the Gelfand transform but has not the vaguest idea what either is good for. Now, at last, he can find out.

The programme of the book is described succinctly and accurately in the preface: "The purpose of this book is to give a mathematical account of Fourier ideas on the circle and the line, on finite commutative groups, and on a few important noncommutative groups. Purely technical aspects of the subject, such as the pointwise convergence of Fourier series of wild functions, will be avoided. The emphasis is placed instead on the extraordinary power and flexibility of Fourier's basic series and integrals and on the astonishing variety of applications in which it is the chief tool." The emphasis on examples and applications is particularly appropriate; they occupy approximately one half of the actual text and constitute the book's cachet, setting it apart from others. We now turn to an actual description of the contents.

Chapter 1, dealing with Fourier series, begins with a rapid introduction to Lebesgue integration, with particular emphasis on the L^2 theory. The Fourier developments of square summable and (later) summable functions are discussed, and a section is devoted to the Gibb's phenomenon. Applications are given to the evaluation of sums, a proof of Wirtinger's inequality, Hurwitz's solution to the isoperimetric problem, Jacobi's identity for theta functions, Weyl's theorem on the equidistribution of arithmetic sequences (the tie-in with statistical mechanics is mentioned), and eigenvalue expansions. Heat flow and wave propagation are discussed in a section on one-dimensional mathematical physics; the heat and wave equations are derived, and there is a lovely application to building a vegetable cellar (it should be 13 feet deep!). After some additional material on eigenfunction expansions, there is a brief treatment of Fourier series in several variables, culminating in a proof of the beautiful theorem of Pólya on return to the origin for random walks in n dimensions.

Chapter 2 is devoted to Fourier integrals. Three different approaches to the Fourier transform and Plancherel theorem for functions in $L^2(R)$ are given, including a development based on eigenvalue expansions in terms of Hermite functions à la Wiener, after which the Fourier inversion formula for $L^1(R)$ is established. Among the applications of this chapter are proofs of the Poisson summation formula, the Euler-MacLaurin summation formula, the central limit theorem, and Heisenberg's inequality (the celebrated uncertainty principle of quantum mechanics). There is also a substantial and enlightening section on band- and time-limited signals and a treatment of Fourier integrals in higher dimensions. Applications of this last topic include a discussion of spherical waves in two and three dimensions, the Radon transform, the multidimensional Poisson summation formula, Minkowski's theorem on convex bodies from the geometry of numbers, and Lord Rayleigh's theorem on random flights.

The third chapter, devoted to the interaction between analytic function theory and Fourier analysis, begins with a short (12 page) course in complex analysis. This covers versions of Cauchy's theorem, the Cauchy integral formula, power series development of analytic functions, the maximum principle, Liouville's theorem, and the Phragmén-Lindelöf principle. Hardy's theorem relating the order of growth on the line of a function and its Fourier transform is proved, and the Paley-Wiener theorem is established. There is also a section dealing with functions on the half-plane of Hardy class H^2. The theory of these functions is then used to treat electrical engineering problems involving filters. Other applications of Fourier transforms include the Wiener-Hopf method (applied to Milne's equation), the remarkable identity of Spitzer, Szegö's theorem, the Müntz-Szász theorem and, finally, the prime number theorem, following Ikehara.

Something of a change of pace occurs in the final chapter, which is entitled "Fourier series and integrals on groups." Beginning with a very abbreviated introduction (or, better, review) of elementary group theory, the authors turn to an explanation of Fourier series and the Fourier transform according to the rubrics of characters, invariant subspaces, eigenfunctions, and homomorphisms. This discussion at once ties together the material of the preceding chapters and sets the stage for the subsequent development; pedagogically, it is altogether admirable. The scene then shifts to finite commutative groups. The fundamental structure theorem for these groups is proved, and their Fourier analysis is quickly obtained, through the Plancherel theorem and the Poisson summation formula. As an application, the authors prove the law of quadratic reciprocity by means of Gaussian sums, whose evaluation (the Landsberg-Schaar identity) is based on Jacobi's identity.

The rest of the chapter is given over to an introduction to the harmonic analysis of noncommutative groups. The development is restricted to the groups $SO(3)$, $SL(2, R)$ and the group of (proper) Euclidean motions of the plane. The treatment here, via spherical harmonics, is quite explicit. The analysis for $SO(3)$ is done in some detail, through the theorem on the representation of L^2 functions; the basic ideas having been developed in this case, the Euclidean motion group and $SL(2, R)$ are discussed more briefly along the same lines. The authors have wisely avoided the temptation to provide a complete treatment; they have offered instead an instructive overview of the subject in three instances of particular geometric interest. The loss of generality is, in this reviewer's opinion, more than compensated by the gain in clarity and perspective. The going here is perhaps rougher than elsewhere in the book, and sticklers for accuracy in notation will bristle at some of the cavalier identifications the authors make, but the careful reader will reap great rewards. And, while specific applications are much less in evidence than in previous sections (there is mention of the Zeeman effect, and Maxwell's interpretation of spherical harmonics as electrostatic potentials is given), the entire chapter constitutes, in a certain sense, a sustained application of the material treated earlier.

Fourier Series and Integrals is written with panache and wit; how else to describe a text that invokes both Wigner and Flanders and Swan on a single page (p. 61)? The style is clear and colloquial; in fact, reading the book is very much like hearing one of the authors lecture. The reader soon learns to put up with, even to savor, certain idiosyncrasies (not to say quirks) of diction: the book is largely written in the second person, and proofs often begin with "bring in" instead of "consider" and end with "as advertised" instead of "as claimed". The level of exposition seems exactly right for the intended audience of advanced undergraduates and beginning graduate students. In particular, the exercises (which play an important role in the development) have been particularly well chosen; the authors have the rare talent of being able to distinguish what needs explanation and what can safely be left to the student. Yet there is no glib handwaving here, no sweeping of ugly details under the rug, nor hiding behind invocatory smokescreens of the hocus locus theorem: everything is out in the open. How delightful to read (of the proof, involving Fubini's theorem, that $L^1(R)$ is an algebra under convolution), "This kind of finicky proof is not very interesting, but it is important to understand precisely what is involved." Quite clearly, the authors know precisely what is involved in Fourier analysis, and what is interesting, and they are not afraid to let the reader in on the secret.

There are a number of minor misprints, as one would expect in a first printing. I noticed the following. Bernoulli is misspelled (p. 2), as is Rogosinski (pp. 282, 285, and 294), Schwartz (pp. 196 and 294), and infimum (pp. 7, 189, and 274). On p. 82, exercise 2, the reference should be to 1.8.3. On p. 99, line -10, an "exp" is missing and on p. 102, line 14, an equals sign should be inequality. The integrand on the bottom line of p. 117 should be $\psi^* A\psi$. Finally, on p. 184, line -12, read "to" for "the" and on p. 210, line 7, "phase" for "phrase." There are doubtless others,[1] equally trivial (but let us get the names right next time).

[1] An error of a rather different sort has recently been pointed out to me by the authors. The constant of p. 44, line -3, and p. 46, line -13, should be 1.178980+. Compare Jeffreys and Jeffreys, *Methods of mathematical physics*, p. 445.

If there is one thing to complain of in this book, it is the almost studied perversity of the notation. Sharps, flats, boldface blobs, bizarre superscripts of every description abound. Who ever heard of using o as a variable of integration? Or of writing $Z(\gamma)$ instead of $\zeta(s)$ for Riemann's zeta function? Or of denoting the convolution of functions by $f \circ g$ instead of $f * g$? (This last will surely throw readers who begin *in medias res.*) What is one to make of an integral whose lower limit is $\square$ and whose upper limit is $\triangle + \alpha$, where $\square$ and $\triangle$ are functions? Sometimes it seems that Dym and McKean have sworn an oath of enmity against typesetters and all other advocates of clean notation. On quite a different line, why do the authors insist on calling the Müntz-Szász theorem the Szász-Müntz theorem? The version they cite is due to Müntz, who, at any rate, precedes Szász both chronologically and lexicographically.

But enough of such quibbles. Without exaggeration, this is surely one of the most important analysis texts of recent years. Dym and McKean have written a remarkable book, one which belongs on the shelf of every analyst and in the hands of every graduate student. They have dedicated it, in part, "To the memory of Terry Mirkil, who started to write a book like this, only better." I knew Terry Mirkil and was a student in the last course on Fourier analysis which he taught. I feel confident that he would have been very proud of this book. But (modulo the trivia detailed above) it is hard for me to see how he, or anyone else, could have done better.

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