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A CHARACTERIZATION OF INNER PRODUCT SPACES

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Let X be a real normed linear space and f a functional on X. Recall that by the first one-sided variation of f at x in the direction h we mean

$$f'_+(x)(h) = \lim_{t \to 0_+} \frac{[f(x + th) - f(x)]}{t}$$

and by the second one-sided variation of f at x in the directions h_1 and h_2 we mean

$$f''_+(x)(h_1, h_2) = \lim_{t \to 0_+} \frac{[f'_+(x + th_1)(h_2) - f'_+(x)(h_2)]}{t}.$$

Let

$$f(x) = \tfrac{1}{2}\|x\|^2 \quad \text{and} \quad (x, y) = f'_+(x)(y).$$

PROPOSITION 1. *Every normed linear space resembles an inner product space in the sense that*

(i) (x, y) *is well defined*;

(ii) $(x, x) \geqq 0$ *with equality if and only if* $x = 0$;

(iii) $\|x\| = (x, x)^{1/2}$.

Moreover, if X is an inner product space with inner product $[\cdot, \cdot]$, *then* $(\cdot, \cdot) = [\cdot, \cdot]$.

PROPOSITION 2. *The following are equivalent*:

(i) X *is an inner product space*;

(ii) (x, y) *is symmetric*;

(iii) (x, y) *is linear in the first variable*;

(iv) $f''_+(0)(x, y)$ *is linear in the first variable.*

The proofs of these and related results will appear elsewhere.

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