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PRIMES WHICH ARE REGULAR FOR ASSOCIATIVE H-SPACES

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This note summarizes some results on associative H-spaces with the homotopy type of a finite CW-complex. Such spaces are called *finite-dimensional* associative H-spaces. The main theorem generalizes a result of Serre [**5**] for the simple classical groups.

A well-known theorem of Hopf [**2**] states that the rational cohomology $H^*(X, Q)$ of a finite-dimensional H-space is an exterior algebra $\Lambda(X_{2N_1-1}, \ldots, X_{2N_K-1})$, where the dimension of X_{2N_i-1} is $2N_i - 1$ and $N_1 \leqq \cdots \leqq N_K$. With this in mind, suppose that X is a finite-dimensional associative H-space with

$$H^*(X, Q) = \Lambda(X_{2N_1-1}, \ldots, X_{2N_K-1}), \quad \text{where } N_1 \leqq \cdots \leqq N_K.$$

Form the space

$$Y = S^{2N_1-1} \times \cdots \times S^{2N_K-1}.$$

One says that a prime p is *regular* for X if there is a function $f\colon Y \to X$ which induces an isomorphism in cohomology with $\boldsymbol{Z}/p\boldsymbol{Z}$ coefficients. The following theorem is due to Serre [**5**].

THEOREM. *If X is a simply connected, simple, compact, connected classical group with rational cohomology as above, then p is regular for X if and only if $p \geqq N_K$.*

Such groups are of the form $\mathrm{SU}(N)$, $\mathrm{Sp}(N)$ or $\mathrm{Spin}(N)$. Serre's proof is a case by case study of these groups. Various generalizations of this theorem have appeared in the literature. Kumpel [**3**], for example, verified that the conclusion of Serre's theorem is true for the exceptional Lie groups, G_2, F_4, E_6, E_7 and E_8. I am announcing a result which generalizes Serre's to finite-dimensional associative H-spaces. Needless to say, certain mild restrictions are necessary. For example, certain assumptions about primitive generation are necessary.

Suppose that X is an H-space with multiplication

$$X \times X \xrightarrow{\mu} X.$$

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Let p_1 and p_2 be the projections

$$X \times X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$$

onto the first and second factors, respectively. One says that an element $x \in H^*(X, R)$ is *primitive* if

$$\mu^*(x) = p_1^*(x) + p_2^*(x).$$

$H^*(X, R)$ is *primitively generated* if it is generated by primitives as an R-algebra.

I can now state the main theorem.

THEOREM I. *Let X be a simply connected associative H-space of finite dimension with*

$$H^*(X, Q) = \Lambda(X_{2N_1-1}, \ldots, X_{2N_K-1}), \quad \text{where } N_1 \leqq \cdots \leqq N_K.$$

Let p be a prime. Then p is regular for X if and only if

(1) $p \geqq N_K$,

(2) *$H^*(X, \mathbf{Z}/p\mathbf{Z})$ is primitively generated,*

(3) *$H^*(X, \mathbf{Z})$ has no p-torsion.*

Larry Smith [**6**] has shown that (1)–(3) are sufficient for p to be a regular prime; so the only remaining point is the necessity of the conditions. Conditions (2) and (3) are immediate and (1) is handled by the following proposition.

PROPOSITION. *With notation as above, if $p < N_K$, p a prime, then p is not regular for X.*

To prove this, one constructs a space $X_{(p)}$ called the localization of X at the prime p (see Sullivan [**9**] for details). One needs the facts that $H^*(X_{(p)}, \mathbf{Z}) \approx H^*(X, \mathbf{Z})_{(p)}$ where the latter denotes the group-theoretic localization (Atiyah and Macdonald [**1**]). If X is an associative H-space then so is $X_{(p)}$ (Mislin [**4**]); and finally, if, given the above notation, p is regular for X, then $X_{(p)} \simeq Y_{(p)}$. The Proposition now follows from

THEOREM II. *If $Y \simeq S^{2N_1-1} \times \cdots \times S^{2N_K-1}$, where $N_1 \leqq \cdots \leqq N_K$ and p is an odd prime less than N_K, then $Y_{(p)}$ is not an associative H-space.*

The proof, which employs Stasheff's notion of an A_p-structure [**7**], is essentially an Adam's operation computation.

The following proposition is a scholium.

PROPOSITION. *Let $\alpha \in \pi_7(BSU(3))$ classify the principal fibration*

$SU(3) \to SU(4) \to S^7$, let E_3 be the induced fibration in

$$\begin{array}{ccccc} SU(3) & = & SU(3) & = & SU(3) \\ \downarrow & & \downarrow & & \downarrow \\ E_3 & \longrightarrow & SU(4) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ S^7 & \xrightarrow{3} & S^7 & \xrightarrow{\alpha} & B(SU(3)) \end{array}$$

then E_3 is not a homotopy associative H-space.

This answers Question 17 in a list of questions presented to the Neuchatel Conference on H-spaces, August 1970 [**8**].

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