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ANALYTICAL CIRCLE GROUP ACTIONS ON COMPACT COMPLEX MANIFOLDS[1]

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1. **Introduction.** Let M be a compact complex manifold (of m complex dimensions), and let G be a compact Lie group acting analytically on M. Then the Dolbeault complexes

$$0 \to \Gamma\left(\overset{p,0}{\wedge}(M)\right) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Gamma\left(\overset{p,q}{\wedge}(M)\right) \to \cdots \to 0,$$

$p = 0, \ldots, m$, are G-elliptic complexes (for the definitions and following notions see [**1**], [**2**], [**3**]) and their analytical indices $\chi(A^{p,*}, G)$ (or simply χ^p) are elements in the group representation ring $R(G)$. Following Hirzebruch [**4**], we have the $\chi_y(A^{p,*}, G)$ (or χ_y)-characteristic, $\sum_{p=0}^{m} \chi^p(-y)^p$ (here we take the alternating sum rather than the sum in [**4**]), which is an element in $R(G)[y]$.

Let $\mathscr{C}_k$ be the category of (M, G) such that M has k fixed points under the analytical action of G, and let $\mathscr{C} = \bigcup_{k=0}^{\infty} \mathscr{C}_k$. In this note we study the category $\mathscr{C}_k$, $k = 2, 3$, for the case $G = S^1$. (Note: (i) $\mathscr{C}_1 = \varnothing$ and (ii) $\chi_y = 0$ for $(M, S^1) \in \mathscr{C}_0$.) Precisely the problem is: what are the necessary conditions for $(M, S^1) \in \mathscr{C}_k$, $k = 2, 3$, and if they do exist, what is their χ_y and the representations of S^1 on the tangent planes over the fixed point set? The main tools for this study are the S^1-index theory and Atiyah-Bott fixed point formula. Only the statement of the result is given here. The details of the proof will appear elsewhere.

2. **Main theorems.**

THEOREM 1. *If $(M, S^1) \in \mathscr{C}$, then $\chi_y \in Z[y]$. Furthermore, if at a fixed point A, the representation of S^1 on the tangent plane T_AM is given by $T_AM(t) = t^{a_1} + \ldots + t^{a_m}$, where $t \in R(S^1) = Z[t, t^{-1}]$, then*

$$(*) \qquad \chi_y = \sum_{S^1(A)=A} \prod_{i=1}^{m} \left(\frac{1 - yt^{a_i}}{1 - t^{a_i}}\right).$$

THEOREM 2. *If $(M, S^1) \in \mathscr{C}_2$, then either* (i) $M = S^2$ *or* (ii) (complex) dim $M = 3$.

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COROLLARY. *Let* $(M, S^1) \in \mathscr{C}_2$.

(i) *If* $M = S^2$, *then* $(*)$ *is given by*

$$1 + y = \frac{1 - yt^a}{1 - t^a} + \frac{1 - yt^{-a}}{1 - t^{-a}}$$

where a is a nonzero integer.

(ii) *If* (complex) dim $M = 3$, *then* $(*)$ *is given by*

$$y + y^2 = \left(\frac{1 - yt^{-a-b}}{1 - t^{-a-b}}\right)\left(\frac{1 - yt^a}{1 - t^a}\right)\left(\frac{1 - yt^b}{1 - t^b}\right) + \left(\frac{1 - yt^{a+b}}{1 - t^{a+b}}\right)\left(\frac{1 - yt^{-a}}{1 - t^{-a}}\right)\left(\frac{1 - yt^{-b}}{1 - t^{-b}}\right)$$

where (a, b) *is a pair of positive integers.*

A simple example for case (i) is given by S^1 acting on S^2 as a rotation along the axis through north and south poles on S^2. It is an interesting problem arising from (ii) that if given any pair (a, b) of positive integers, can we find an analytical S^1-action of case (ii) type?

THEOREM 3. *If* $(M, S^1) \in \mathscr{C}_3$, *then M must be a complex surface and* $(*)$ *is given by*

$$1 + y + y^2 = \left(\frac{1 - yt^a}{1 - t^a}\right)\left(\frac{1 - yt^b}{1 - t^b}\right) + \left(\frac{1 - yt^{a-b}}{1 - t^{a-b}}\right)\left(\frac{1 - yt^{-b}}{1 - t^{-b}}\right) + \left(\frac{1 - yt^{b-a}}{1 - t^{b-a}}\right)\left(\frac{1 - yt^{-a}}{1 - t^{-a}}\right)$$

where $a \neq b$ *are any nonzero integers.*

EXAMPLE. The linear action of S^1 on $CP(2)$ given by: $(z_0, z_1, z_2) \to (z_0, t^a z_1, t^b z_2)$, $a \neq b$, belongs to $\mathscr{C}_3$.

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