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# INVARIANT SUBSPACES OF $L^\infty$ AND $H^\infty$

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Let $T$ be the unit circle, and let $L^\infty$ and $H^\infty$ be the usual spaces of bounded functions. Let $R$ be the group of rotations $z \mapsto e^{i\lambda}z$ and let $M$ be the Möbius group

$$z \mapsto e^{i\lambda}\frac{z - z_0}{1 - \bar{z}_0 z}.$$

Let $R$ and $M$ act on $L^\infty$ by substitution.

THEOREM 1. *Let $F$ be a closed $M$-invariant subspace of $L^\infty$, with $z \in F$ and $zF \subseteq F$. Then $F$ does not properly contain any closed $M$-invariant subspaces of finite codimension.*

Examples of such subspaces $F$ are $F = L^\infty$, $F = H^\infty$, $F = A$, $F = C(T)$, $F = \mathscr{R}$, and $F = \mathscr{R}_h$. Here, $A$ is the disc algebra, $\mathscr{R}$ is the space of functions in $H^\infty$ that have radial limits along every radius, and $\mathscr{R}_h$ is the space of functions in $H^\infty$ for which the radial limit fails to exist at most on a set of $e^{i\theta}$ of Hausdorff $h$-measure 0.

THEOREM 2. *There exists an $R$-invariant closed hyperplane in $L^\infty$ that contains the space $C(T)$ but does not contain $H^\infty$.*

COROLLARY. *There exists an $R$-invariant closed hyperplane in $H^\infty$ that contains $A$.*

THEOREM 3. *Let $B$ be a closed $R$-invariant subspace of $H^\infty$ with $B \supseteq \mathscr{R}_h$ such that either*

(i) *$B/\mathscr{R}_h$ is separable or*

(ii) *$B$ is a countably generated $\mathscr{R}_h$ module.*

*Then $B = \mathscr{R}_h$.*

The proofs, especially of Theorem 1, are long, and we will give the details in a subsequent paper, giving here only an outline of the main steps in the proof of Theorem 1.

To begin with, we remark that $M$ is isomorphic to PSL(2, $\boldsymbol{R}$), which is

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SL(2, $\boldsymbol{R}$) modulo its center. We suppose that $F \supseteq E$, where $E$ is closed and $M$-invariant and $\dim F/E < \infty$.

LEMMA 1. *A bounded, finite dimensional representation of* SL(2, $\boldsymbol{R}$) (*in the discrete topology*) *must be trivial.*

LEMMA 2. *If $F \neq E$ then $F$ contains a closed $M$-invariant subspace $E'$ such that* $\dim F/E' = 1$.

From now on, we will suppose that $\dim F/E \leqq 1$, and conclude that $F = E$.

LEMMA 3. *For any $f \in F$ and $\mu \in M, f - f \circ \mu \in E$.*

LEMMA 4. $E \supseteq A$.

DEFINITION. The function $f \in L^\infty$ is $M$-analysable, and we write $f \in \mathfrak{A}_M$, if there is a complex constant in the norm-closed convex hull of the orbit of $f$ under $M$.

LEMMA 5. $E \supseteq \mathfrak{A}_M \cap F$.

LEMMA 6. *If $f$ is continuous at one point $z_0 \in T$, then $f \in \mathfrak{A}_M$.*

LEMMA 7 (*trivial*). *Any $f \in L^\infty$ may be written $f = f_1 + f_2$ where $f_1$ is continuous at $+1$ and $f_2$ is continuous at $-1$.*

The combination of Lemmas 3–7 implies that $F = E$.

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