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COBORDISM OF $U(n)$-ACTIONS

BY CONNOR LAZAROV AND ARTHUR WASSERMAN[1]

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0. **Introduction.** Let G be a compact Lie group acting on a C^∞ manifold. A G invariant stable complex structure on M is a complex structure J on $T(M) \oplus \varepsilon^r$ (where ε^r is a trivial bundle) such that for each g in G, dg $\oplus$ id commutes with J. We will be concerned with the case where $G = U(n)$ and the action is free or regular. We will study the resulting bordism theories.

DEFINITION. Let M be a compact $U(n)$ manifold. M is called a regular $U(n)$ manifold if

1. Every isotropy group is conjugate to $U(k)$ for some $0 \leqq k \leqq n$.
2. For some r, $T(M) \oplus \varepsilon^r$ has a $U(n)$ invariant complex structure J such that the representation of the isotropy group $U(n)_x$ at $(T(M) \oplus \varepsilon^r)_x$ is equivalent to a sum of copies of the standard complex representation of $U(n)_x$ plus a trivial complex representation. (Remark. If $U(n)_x = g^{-1}U(k)g$, then $U(n)_x$ acts in the obvious way on $g^{-1}C^k \subset C^n$ and this is the standard representation.)

We define homotopy and equivalence classes of such structures analogously to [2]. The resulting bordism theory is denoted by $\Omega U(n)_*$. We denote the bordism theory of free $U(n)$-actions by $\Omega_*^{(n)}$. The main results are summarized in the following theorem.

THEOREM. *$\Omega_*^{(n)}$ and $\Omega U(n)_*$ are free MU_* modules. Any connected regular $U(n)$ manifold on which $U(n)$ acts nontrivially is bordant in $\Omega U(n)_*$ to a regular $U(n)$ manifold in which every isotropy group is conjugate to $U(1)$ or $U(0)$.*

Warning. $\Omega_*^{(n)}$ is not obviously $MU_*(BU(n))$.

1. **Relation between $\Omega_*^{(n)}$ and $\Omega U(n)_*$.** As in [3], [7] we construct a long exact sequence $\to D^{*,i} \to D^{*,i-1} \to E^{*,i-1} \to D^{*,i} \to \cdots$ and a resulting exact couple and a spectral sequence. Then E^∞ is associated to a filtration of $\Omega U(n)_*$. For $k \neq n$, $E^1_{*,k}$ is the bordism group of pairs (E, M) where E is a complex $U(n)$ vector bundle over the regular $U(n)$ manifold M such that every point in M has isotropy group conjugate to $U(n-k)$ and the representation of $U(n)_x$ on E_x is a sum of copies of the standard complex representation of $U(n)_x$. The pair (E, M) is completely determined by the $U(n-k) \times U(k)$ manifold M_0, the points in M with isotropy group

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$U(n-k)$, and the vector bundle $E|_{M_0} = F \otimes \rho_{n-k}$ where ρ_{n-k} is the standard complex representation of $U(n-k)$ and F is a free $U(k)$ complex vector bundle over the free $U(k)$ manifold M_0. (F, M_0) completely determines (E, M) so $E^1_{*,k} = \bigoplus_p \Omega^{(k)}_*(BU(p))$. The differential $d: E^1_{*,k} \to E^1_{*,k+1}$, $k+1 \neq n$, takes (F, M_0) to $(H \times_{U(1) \times U(k)} U(k+1), S(F) \times_{U(1) \times U(k)} U(k+1))$ where H is the canonical hyperplane bundle over the sphere bundle $S(F)$. $E^1_{*,n} = \Omega^{(n)}_*$ and the differential takes (F, M_0) in $E^1_{*,n-1}$ to $S(F) \times_{U(1) \times U(n-1)} U(n)$.

2. **The theories $\Omega^{(k)}_*$.** The bordism theory of stably complex free $U(k)$ manifolds is the same as the bordism theory gotten from pairs (P, M), where P is a principal $U(k)$ bundle, together with a complex structure on the vector bundle $\nu_M \oplus \mathrm{Ad}(P)$. Here ν_M is the stable normal bundle and $\mathrm{Ad}(P) = P \times_{U(k)} R^{k^2}$ with $U(k)$ acting on R^{k^2} via the adjoint representation. We construct spaces $B\,\mathrm{Ad}_{k,n}$ by taking the pullback of $BU(N)$ via

$$B(\mathrm{id} \times \mathrm{Ad}): BO(2n) \times BU(k) \to BO(2N) \tag{2.1}$$

where $2N = 2n + k^2$ or $2n + k^2 + 1$. $M\,\mathrm{Ad}_{k,n}$ is the Thom space of the oriented vector bundle pulled up from the universal bundle over $BO(2n)$. There are obvious maps $S^2 \wedge M\,\mathrm{Ad}_{k,n} \to M\,\mathrm{Ad}_{k,n+1}$ and so we obtain a spectrum $M\,\mathrm{Ad}_k$. It is clear that $\pi_*(M\,\mathrm{Ad}_k) = \Omega^{(k)}_*$. If we replace $U(k)$ by T^k, the maximal torus, in this procedure we get spaces $B'\mathrm{Ad}_{k,n}$, $M'\mathrm{Ad}_{k,n}$. $B'\mathrm{Ad}_{k,n} \to B\,\mathrm{Ad}_{k,n}$ is a $U(k)/T^k$ bundle. The representation $\mathrm{id} \times \mathrm{Ad}$ induces a map

$$\varphi: BU(n) \times BT^k \to B'\mathrm{Ad}_{k,n}. \tag{2.2}$$

LEMMA (2.3). *φ^* is an isomorphism in cohomology in dimensions $< 2n-1$.*

Now let $P' = E_{O(2n) \times U(k)} \times_{U(n) \times T^k} O(2N)$, $P = E_{O(2n) \times U(k)} \times_{U(n) \times U(k)} O(2N)$ and V' and V the corresponding complex vector bundles over $P'/U(N) = B'\mathrm{Ad}_{k,n}$ and $P/U(N) = B\,\mathrm{Ad}_{k,n}$. Let c'_j denote the Chern classes of V and V'. Using some representation theory and (2.3) we show that

LEMMA (2.4). *$\varphi^*(c'_j) = c_j + \sum c_{j-q}(t_{i_1} - t_{l_1}) \cdots (t_{i_q} - t_{l_q})$, where c_j are the Chern classes in $H^*(BU(n))$, $t_1, \ldots, t_k$ are the generators for $H^*(BT^k)$.*

$$H^*(B\,\mathrm{Ad}_{k,n}) = Z[c'_1, c'_2, \ldots, \sigma_1(t), \ldots, \sigma_k(t)]$$

through dimension $2n-2$. $\sigma_j(t)$ is the symmetric function of $t_1, \ldots, t_k$.

LEMMA (2.5). *$H^*(M\,\mathrm{Ad}_k)$ is a free Z module with basis $U' \cup S_I(c') \cup S_J(\sigma)$ where U' is the "universal" Thom class for $M\,\mathrm{Ad}_k$, I runs through all finite sequences of nonnegative integers, and J runs through sequences $(j_1, \ldots, j_k)$.*

There is an obvious map $BU(m) \times B\,\mathrm{Ad}_{k,n} \to B\,\mathrm{Ad}_{k,n+m}$ which gives rise to a map

$$MU \wedge M\,\mathrm{Ad}_k \to M\,\mathrm{Ad}_k. \tag{2.6}$$

Thus $\Omega_*^{(k)}$ becomes an MU_* module (which is evident geometrically) and $H^*(M\,\mathrm{Ad}_k)$ becomes an $H^*(MU)$ comodule. We exploit this comodule structure together with Milnor's results **[4]** on $H^*(MU; Z_p)$ to obtain

THEOREM (2.7). *For each prime p, $H^*(M\,\mathrm{Ad}_k; Z_p)$ is a free module over $A_p/(Q_0)$ with basis $U' \cup S_\lambda(c') \cup S_J(\sigma)$, where λ runs over all finite sequences containing no $p^j - 1$, A_p is the mod p Steenrod algebra, and (Q_0) is the two-sided ideal generated by the Bockstein.*

It follows that the Adams spectral sequence for $\pi_*(M\,\mathrm{Ad}_k)$ collapses and $\Omega_*^{(k)}$ is torsion-free. Let Y_w be the Milnor manifolds described in **[6]** which form a basis for MU_*. Let $0 \leqq i_1 \leqq \cdots \leqq i_k$ be integers. Consider $Y_w \times CP^{i_1} \times \cdots \times CP^{i_k} = Y_w \times CP^I$. These elements represent elements in $MU^*(BT^k) = \pi_*(MU \wedge BT^k)$. From (2.2) we get maps

$$\pi_*(MU \wedge BT^k) \to \pi_*(M'\mathrm{Ad}_k) \to \pi_*(M\,\mathrm{Ad}_k). \tag{2.8}$$

Let $\mathrm{Ad}(Y_w \times CP^I)$ represent the image of the elements $Y_w \times CP^I$ in $\pi_*(M\,\mathrm{Ad}_k)$. Using an argument similar to **[4]**, **[6]** for MU_* we obtain

THEOREM (2.9). *$\Omega_*^{(k)}$ is a free Z module on* $\mathrm{Ad}(Y_w \times CP^I)$. *Thus the stably complex $U(k)$ manifolds $S^{2i_1-1} \times \cdots \times S^{2i_k-1} \times_{T^k} U(k) = S^I \times_{T^k} U(k)$ form a free MU_* basis for $\Omega_*^{(k)}$.*

COROLLARY (2.10). *The homomorphism $\Phi: MU_*(BU(k)) \to \Omega_*^{(k)}$ taking CP^I to $S^I \times_{T^k} U(k)$ is an isomorphism of MU_* modules.*

3. **Application to $\Omega U(n)_*$.** There is an obvious pairing $\Omega_*^{(k)} \otimes_{MU_*} MU_*(X) \to \Omega_*^{(k)}(X)$ and the composition

$$\begin{aligned} MU_*(BU(k) \times X) &= MU_*(BU(k)) \underset{MU_*}{\otimes} MU_*(X) \\ &\xrightarrow{\Phi \times \mathrm{id}} \Omega_*^{(k)} \underset{MU_*}{\otimes} MU_*(X) \to \Omega_*^{(k)}(X) \end{aligned} \tag{3.1}$$

is a natural transformation which is an isomorphism for $X =$ point, hence for all X. The map $\Omega_*^{(k)}(BU(p)) \xrightarrow{d} \Omega_*^{(k+1)}(BU(p-1))$ of §2 gives rise to a map

$$MU_*(BU(k) \times BU(p)) \xrightarrow{d} MU_*(BU(k+1) \times BU(p-1)). \tag{3.2}$$

THEOREM (3.3). *The sequence*

$$\to MU_*(BU(k) \times BU(p)) \xrightarrow{d} MU_*(BU(k+1) \times BU(p-1)) \to$$

is a split exact sequence of MU_* modules.

INGREDIENTS OF PROOF. From [3] we know the cohomology sequence

$$\to H_*(BU(k) \times BU(p)) \xrightarrow{d} H_*(BU(k+1) \times BU(p-1)) \to$$

is a split exact. d is $i_*\pi^\natural$ where $BU(k) \times BU(1) \times BU(p-1) \xrightarrow{\pi} BU(k) \times BU(p)$ and $BU(k) \times BU(1) \times BU(p) \xrightarrow{i} BU(k+1) \times BU(p-1)$. The Thom homomorphism $\mu: MU_*(BU(k) \times BU(p)) \to H_*(BU(k) \times BU(p))$ commutes with the two d's. Using the collapsing of the Atiyah spectral sequence $E^2 = H_*(BU(k) \times BU(p)) \otimes MU_*$, an argument similar to [**1**, 18.1], the fact that d^2 must be zero, and induction, the result follows.

COROLLARY (3.4). $\to \Omega^{(k)}(BU(p)) \xrightarrow{d} \Omega^{(k+1)}(BU(p-1)) \to$ *is exact.*

From this the main theorem follows by arguments identical to [**3**].

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DEPARTMENT OF MATHEMATICS, HERBERT H. LEHMAN COLLEGE (CUNY), BRONX, NEW YORK 10468

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104 (Current address of Arthur Wasserman)

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540