BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 4, July 1972

INDUCED REPRESENTATIONS OF C^*-ALGEBRAS

BY MARC A. RIEFFEL[1]

Communicated by Calvin C. Moore, December 2, 1971

In this announcement we will indicate how Mackey's definition [7] of induced representations of locally compact groups can be generalized to the setting of C^*-algebras, and how the imprimitivity theorem [8] can be formulated in this setting. Proofs, as well as discussion of other theorems in the theory of induced representations, will appear elsewhere.

Let G be a locally compact group, and let H be a closed subgroup of G. Let $C_c(G)$ and $C_c(H)$ denote the algebras (under convolution) of continuous complex-valued functions of compact support on G and H respectively, viewed as dense involutory subalgebras of the group C^*-algebras [5] of G and H. Then elements of $C_c(G)$ can be convolved on the right by elements of $C_c(H)$, and under this action $C_c(H)$ acts as an algebra of right centralizers [6] on $C_c(G)$.

Let Δ and δ denote the modular functions of G and H respectively, and let γ be the function on H defined by

$$\gamma(s) = (\Delta(s)/\delta(s))^{1/2}$$

for $s \in H$. Let P denote the linear map from $C_c(G)$ onto $C_c(H)$ defined by

$$P(f)(s) = \gamma(s)f(s)$$

for $f \in C_c(G)$ and $s \in H$. Then P commutes with the involutions. Furthermore, a reformulation of a theorem of Blattner [4] says that P is a positive map, in the sense that $P(f^* * f)$ is a positive element of the pre-C^*-algebra $C_c(H)$ for all $f \in C_c(G)$. Now let the right action of $C_c(H)$ on $C_c(G)$ be redefined by $f \cdot \phi = f * (\gamma\phi)$ for $f \in C_c(G)$ and $\phi \in C_c(H)$, where $\gamma\phi$ denotes the pointwise product of γ and ϕ. Under this new action $C_c(H)$ still acts as an algebra of right centralizers on $C_c(G)$, but now P satisfies the conditional expectation property

$$P(f \cdot \phi) = P(f) * \phi$$

for $f \in C_c(G)$ and $\phi \in C_c(H)$. In general P is not norm-continuous. However, P is relatively bounded, in the sense that for any $g \in C_c(G)$ the map

---

AMS 1970 *subject classifications*. Primary 22D30, 46K10, 46L05; Secondary 16A64, 20C15, 43A95, 46H25, 46M15.

[1] The research reported in this announcement was conducted while I was visiting at the University of Pennsylvania last year. I would like to thank the members of the Department of Mathematics of the University of Pennsylvania for their warm hospitality during my stay there. Special thanks go to J. M. G. Fell, with whom I had many extremely helpful conversations about this research. My work on this research was partially supported by National Science Foundation grant GP-25082.

$f \mapsto P(g^* * f * g)$ is norm-continuous. Finally, products of elements of $C_c(G)$ are dense in $C_c(G)$ with respect to the norm $f \mapsto (\|P(f^* * f)\|_{C^*(H)})^{1/2}$. These considerations lead us to make:

DEFINITION 1. Let A and B be pre-C^*-algebras, with B acting as right centralizers on A. By a *generalized conditional expectation* from A to B we mean a linear map, P, from A to B satisfying

(1) $P(a^*) = P(a)^*$ for $a \in A$.
(2) $P(a^*a) \geqq 0$ for $a \in A$.
(3) $P(ab) = P(a)b$ for $a \in A$ and $b \in B$.
(4) For all $c \in A$ the map $a \mapsto P(c^*ac)$ from A to B is continuous.
(5) For every $a \in A$ and $\varepsilon > 0$ there is a $c \in A^2$ such that $\|P((a - c)^*(a - c))\|_B < \varepsilon$.
(6) The range of P generates B.

If P is a generalized conditional expectation from A to B, and if V is a Hermitian B-module (that is, the Hilbert space of a continuous non-degenerate $*$-representation of B), then on the algebraic tensor product $A \otimes_B V$ a pre-inner-product can be defined whose value on elementary tensors is given by

$$\langle a_1 \otimes v_1, a \otimes v \rangle = \langle P(a^*a_1)v, v_1 \rangle.$$

Furthermore, the obvious left action of A on $A \otimes_B V$ gives a continuous $*$-representation of A by operators which are bounded with respect to this pre-inner-product, and this representation is nondegenerate on the corresponding Hilbert space.

DEFINITION 2. The Hermitian A-module obtained from the action of A on the space $A \otimes_B V$ equipped with the pre-inner-product indicated above is called the *Hermitian A-module obtained by inducing V from B to A via P.*

It is not difficult to show that the induced modules constructed as above via the generalized conditional expectation associated earlier to a locally compact group and a closed subgroup coincide with Mackey's induced representations as generalized to the possibly nonseparable case by Blattner [2].

The imprimitivity theorem describes which Hermitian A-modules are obtained by inducing Hermitian B-modules up to A via P. To formulate the theorem we first define a pre-C^*-algebra of operators on A. For simplicity we assume that P is faithful (as it is in the group case), that is, that if $P(a^*a) = 0$ then $a = 0$. Define a B-valued inner-product on A by

$$\langle a_1, a \rangle_B = P(a^*a_1)$$

for $a, a_1 \in A$. Then the appropriate definition of the algebra, $L(A)$, of "bounded" operators on A for this B-valued inner-product is the set of operators, T, from A into itself satisfying

(1) There is a constant, k, such that

$$\langle Ta, Ta\rangle_B \leqq k^2\langle a, a\rangle_B$$

as positive elements of the pre-C^*-algebra B, for all $a \in A$.

(2) There is an operator, T^*, from A into itself which satisfies condition (1) above, and for which

$$\langle Ta, a_1\rangle_B = \langle a, T^*a_1\rangle_B$$

for all $a, a_1 \in A$.

The norm of T is defined to be the least constant k for which condition (1) holds. Then $L(A)$ is a pre-C^*-algebra. Furthermore, with the obvious action of $L(A)$ on $A \otimes_B V$, every Hermitian A-module induced from B is seen to be in fact a Hermitian $L(A)$-module, thus giving a necessary condition for a module to be induced.

To obtain a condition which is also sufficient, we define the analogue for $L(A)$ of the two-sided ideal of compact, or rather finite rank, operators on a Hilbert space. This will be just the linear span, E, of the "rank one" operators $T_{(c,d)}$ for $c, d \in A$, where

$$T_{(c,d)}a = c\langle a, d\rangle_B$$

for all $a \in A$.

THE IMPRIMITIVITY THEOREM. *A Hermitian A-module W is induced from some Hermitian B-module via P if and only if W can be made into a Hermitian E-module in such a way that $a(ew) = (ae)w$ for all $a \in A$, $e \in E$ and $w \in W$ (where ae denotes the product in $L(A)$, which will again be an element of E).*

The imprimitivity theorem of Mackey [**8**], as generalized to the possibly nonseparable case by Blattner [**4**], can be derived from the above theorem.

Actually, A as an E-B-bimodule is an analogue of the "invertible bimodules" which occur in the Morita theorems [**9**] (see [**1**, p. 60]), and we have

THEOREM. *The category of Hermitian E-modules is equivalent to the category of Hermitian B-modules.*

This generalizes Theorem 6.4 of [**8**] as generalized to the possibly nonseparable case in [3]. Such analogues of "invertible bimodules" over pre-C^*-algebras can be described abstractly.

REFERENCES

1. H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR **40** #2736.

2. R. J. Blattner, *On induced representations*, Amer. J. Math. **83** (1961), 79–98. MR **23** #A2757.

3. ———, *On a theorem of G. W. Mackey*, Bull. Amer. Math. Soc. **68** (1962), 585–587. MR **25** #5135.

4. ———, *Positive definite measures*, Proc. Amer. Math. Soc. **14** (1963), 423–428. MR **26** #5095.

5. J. Dixmier, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **30** #1404.

6. B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. (3) **14** (1964), 299–320. MR **28** #2450.

7. G. W. Mackey, *Induced representations of locally compact groups*, I, Ann. Math. (2) **55** (1952), 101–139. MR **13**, 434.

8. ———, *Unitary representations of group extensions*. I, Acta Math. **99** (1958), 265–311. MR **20** #4789.

9. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142. MR **20** #3183.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720