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EXTENSIONS OF MAPS IN SPACES WITH PERIODIC HOMEOMORPHISMS

BY JAN W. JAWOROWSKI

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ABSTRACT. This is an announcement of some results on extension and retraction properties in the equivariant category of compact metric spaces with periodic maps of a prime period. If X and Y are spaces in this category, A is an equivariant closed subspace of X and $f:A \to Y$ is an equivariant map then the existence of an extension of f does not, in general, imply the existence of an equivariant extension. In the case, however, when A contains the fixed point set of the periodic map and $\dim (X - A) < \infty$, a condition for the existence of an extension is also sufficient for the existence of an equivariant extension. In particular, if follows that a finite-dimensional space Y in this category is an equivariant AR (resp. equivariant ANR) iff both Y and the fixed point set of the periodic map are AR's (resp. ANR's).

1. **Introduction.** Let us consider spaces with operators from Z_p, the cyclic group of order p; that is to say, pairs (X, a), where X is a space and $a:X \to X$ is a periodic homeomorphism of period p, $a^p = 1$. Such objects form a category which we denote by $\mathscr{A}_p$; a morphism $(X, a) \to (Y, b)$ in $\mathscr{A}_p$ is a map $f:X \to Y$ which is consistent with the homeomorphisms $a:X \to X$ and $b:Y \to Y$; it is also called an equivariant map. If (X, a) is an object of $\mathscr{A}_p$ and $Z \subset X$ is such that $aZ = Z$ then the periodic map a defines a periodic map $Z \to Z$ which sometimes will be denoted by a_Z; and (Z, a_Z) will be called an equivariant subspace of (X, a).

Some general extension theorems for spaces with operators exist in the literature, such as Gleason [3] and Palais [10]. Such theorems are generally some forms of the Tietze Extension Theorem carried over to an equivariant category. That is, they state that certain objects Y are injective, i.e. have the property that given any object X (suitably restricted) an equivariant closed subobject $A \subset X$ and an equivariant map $f: A \to Y$, there exists an equivariant extension $g: X \to Y$ of f over X. Such a space Y may be also called an absolute extensor in the category in question or an "equivariant absolute extensor" (EAE). Similarly, one can use the concepts of "equivariant absolute neighborhood extensor" (EANE), "equivariant absolute retract" (EAR) and "equivariant absolute neighborhood retract" (EANR) in a given equivariant category. The given group G of operators is usually assumed to be an orthogonal or a linear group and Y is assumed to be an equivariant convex subset of a vector space on which G acts linearly. Thus, for instance, the Tietze-Gleason-Palais Theorem asserts

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that a Euclidean space E is an EAE for an orthogonal group (and for normal spaces). Similarly, one can prove a Dugundji Equivariant Theorem in a suitable category.

The assumptions of convexity of Y and linearity of the action allows one to obtain such an extension theorem directly from the Tietze (resp. Dugundji) extension theorem. These two assumptions are closely related: the action of G can generally be linearized (see [**1**], [**9**] and [**7**]); but then the convexity of Y may be distorted.

The convex-linear assumption is too restrictive in the topological case. Thus, for instance, the above-mentioned extension theorems cannot be used to answer the following question: Let E be a Euclidean space and let Y be an equivariant compact subset of $E \times E$ (with respect to the diagonal symmetry $(x, y) \to (y, x)$) such that Y is an AR; hence there exists a retraction $E \times E \to Y$. Under what assumptions does there exist an equivariant retraction $E \times E \to Y$?

It will be seen from the remarks below that an equivariant retraction may not, in general, exist. We shall establish, in fact, necessary and sufficient conditions for the existence of an equivariant retraction in the finite-dimensional case and for the group $G = \boldsymbol{Z}_p$, p prime, acting on compact metric spaces.

Detailed proofs will appear in [**4**] and [**5**].

2. **Equivariant retractions and Floyd's example.** Any object (X, a) of $\mathscr{A}_p$, i.e. a compact metric space X with a periodic map $a : X \to X$ of period p, can be equivariantly embedded in a finite-dimensional cube or a Hilbert cube Q with a linear, even an isometric, periodic map $Q \to Q$ (this is a particular case of a linearization process used in [**1**]). We choose any embedding $X \subset Q$ and define an equivariant embedding $h : X \to Q^p = Q \times \cdots \times Q$ (p times) by $x \to (x, ax, \ldots, a^{p-1}x)$. Then the periodic map becomes a cyclic permutation of the coordinates.

Suppose then that Y is an equivariant closed subset of (Q, a), where Q is a cube (a finite-dimensional or a Hilbert cube) with a linear periodic map $a : Q \to Q$ of period p. Let $F(a)$ and $F(a_Y)$ denote the fixed point sets of a on Q and Y, respectively. If there is an equivariant retraction $r : Q \to Y$ of Q to Y then r defines a retraction of $F(a)$ to $F(a_Y)$. But since a is linear, the fixed point set $F(a)$ is convex and hence an AR (a cube in fact; see [**6**] or [**8**]). It follows that then both Y and the fixed point set $F(a_Y)$ have to be AR's. There is, however, an example due to E. E. Floyd [**2**] of a compact AR Y (which is, in fact, a contractible 5-dimensional polyhedron) with an involution $a : Y \to Y$ whose fixed point set is not contractible: $H_3(F(a_Y); \boldsymbol{Z}_3) \neq 0$. One can also construct an example of a compact AR with an involution $a : Y \to Y$ such that $F(a_Y)$ is not an ANR.

This suggests the following question: suppose that (Y, a) is an object of $\mathscr{A}_p$ such that both Y and the fixed point set $F(a)$ are AR's (resp. ANR's). Is then (Y, a) an EAR (resp. an EANR)?

Our main theorem (§3) will imply that the answer to the above question is "Yes", provided that $\dim Y < \infty$.

3. **Equivariant extension theorem.**

(3.1) THEOREM. *Let (X, a) be an object of $\mathscr{A}_p$, p prime, and let A be an equivariant closed subspace of X containing the fixed point set $F(a)$ of a and such that* $\dim(X - A) < \infty$. *Let (Y, b) be an object of $\mathscr{A}_p$ and let $f: A \to Y$ be an equivariant map. Then*

(i) *if Y is an AR, there exists an equivariant extension $g: X \to Y$ of f over X;*

(ii) *if Y is an ANR, there exists an equivariant extension $g: U \to Y$ of f over an equivariant neighborhood U of A in X.*

Thus this is an equivariant extension theorem in which the assumptions on Y are topological, rather than geometric, in nature. For the proof see [**4**]. It is an open question whether the finite-dimensional assumption, $\dim(X - A) < \infty$, in this theorem is essential.

4. **Equivariant absolute retracts.** The extension theorem of §3 can now be used to answer the question stated in §2 as follows:

(4.1) THEOREM. *Let (Y, a) be an object of $\mathscr{A}_p$ such that* $\dim Y < \infty$. *Then*

(i) *Y is an EAR iff both Y and the fixed point set $F(a_Y)$ are AR's.*

(ii) *Y is an EANR iff both Y and the fixed point set $F(a_Y)$ are ANR's.*

The fact that the conditions of (i) and (ii) are sufficient follows easily from the extension Theorem (3.1). Since Y is finite-dimensional, it suffices to show that Y is an equivariant retract (resp. equivariant neighborhood retract) of an n-cube (I^n, a) with a linear periodic map $a: I^n \to I^n$. A retraction (resp. neighborhood retraction) can be obtained by combining a retraction provided by the Theorem (3.1) with a retraction of $F(a)$ (or of a neighborhood) to $F(a_Y)$. For details see [**4**].

As an illustration we mention the following corollary:

(4.2) COROLLARY. *Let E be a Euclidean space and let X be an equivariant compact subset of $E \times E$, with respect to the diagonal symmetry $(x, y) \to (y, x)$. Let F be the diagonal of $E \times E$. Then X is an equivariant retract of $E \times E$ (resp. an equivariant neighborhood retract in $E \times E$) iff both X and $X \cap F$ are AR's (resp. ANR's).*

Again, the question whether (4.1) holds without the finite-dimensional restriction is open. In particular, we can ask the following question.

(4.3) QUESTION. *Let Q be a Hilbert cube and let X be subset of $Q \times Q$ which is symmetric with respect to the diagonal symmetry $(x, y) \to (y, x)$, with the diagonal F. Suppose that X is a retract of $Q \times Q$ and $X \cap F$ is a retract of F. Does there exist a symmetric retraction of $Q \times Q$ to X?*

5. **Addition theorems and examples of equivariant retracts.** Let (Q, a) be a linear Hilbert cube in $\mathscr{A}_p$, i.e. a Hilbert cube with a linear periodic map $a: Q \to Q$ of period p. Then any equivariant convex and closed subset of Q is an EAR. The following addition theorem enables us to produce more examples of EAR's and EANR's:

(5.1) THEOREM. *Let (Q, a) be a linear Hilbert cube in $\mathscr{A}_p$ and X be a closed subset of Q. If, for any nonempty subset S of $\{0, 1, \ldots, p-1\}$, the intersection*

$$\bigcap_{i \in S} a^i X$$

is an AR (resp. an ANR) and $X \cap (aX) \cap \cdots \cap (a^{p-1}X)$ is an EAR (resp. an EANR), then the union $X \cup (aX) \cup \cdots \cup (a^{p-1}X)$ is an EAR (resp. an EANR).

The proof of (5.1) is completely elementary and mostly combinatorial, but quite lengthy (see [**5**]).

(5.2) COROLLARY. *If (Q, a) is a linear Hilbert cube in $\mathscr{A}_p$ and X is a closed convex subset of Q, then $X \cup (aX) \cup \cdots \cup (a^{p-1}X)$ is an EANR. If, moreover, $X \cap (aX) \cap \cdots \cap (a^{p-1}X)$ is nonempty, then $X \cup (aX) \cup \cdots \cup (a^{p-1}X)$ is an EAR.*

By using another addition theorem similar to (5.1) we can obtain the following fact (see [**5**]):

(5.3) COROLLARY. *If (Q, a) is a linear Hilbert cube in $\mathscr{A}_p$ and X is an equivariant subset of Q which is the union of a finite collection of closed convex subsets of Q, then X is an EANR.*

Thus, in particular, any compact polyhedron with a simplicial periodic map is an EANR. This fact can also be deduced from (3.1).

6. **Nonlinear involutions on cubes.** Many examples on nonlinear involutions exist in the literature. The example of Floyd mentioned in §2 can be used to construct an involution $a: Q \to Q$ of the Hilbert cube Q whose

fixed point set is not an AR as follows: Let Y be a contractible polyhedron of Floyd with an involution $a:Q \to Q$ such that $F(a)$ is not an AR. Let $c = a \times 1_Q : Y \times Q \to Y \times Q$. By a result of James E. West [**11**], $Y \times Q \cong Q$, since Y is a contractible polyhedron. Thus $c:Q \to Q$ is an involution on Q such that (Q, c) is not an EAR.

Kyung W. Kwun described to the author a construction of an involution on I^n, $n \geqq 5$, whose fixed point set is not an AR.

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Department of Mathematics, Indiana University, Bloomington, Indiana 47401