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ANALYTIC FUNCTIONS WITH UNIVALENT DERIVATIVES AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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ABSTRACT. Functions f, analytic and univalent in the unit disc, and such that all successive derivatives $f^{(k)}$ are univalent in this disc, are necessarily transcendental entire functions of exponential type. These functions, and functions f having an infinite number of derivatives $f^{(n_k)}$ univalent in the unit disc, are discussed. Entire functions of bounded index are of exponential type and their properties are also discussed.

1. **Introduction.** Let $f(z)$ be analytic in the unit disc $D:|z| < 1$. We say that f is univalent in D if for each pair of distinct points z_1, z_2 in D, $f(z_1) \neq f(z_2)$. In §§1–4 we give a brief survey of functions analytic and[1] univalent in D. Functions f such that $f(z)$ and each successive derivative $f^{(k)}(z)$ are univalent in D are considered next in §5. Such functions f must be transcendental entire functions of exponential type. Related problems of functions f such that $f(z)$ and a sequence of derivatives $f^{(n_k)}(z)$ are univalent or of functions f such that $f(z)$ is entire and $f^{(k)}(z)$ is univalent in $|z| < \rho_k$ $(\rho_k > 0)$ are considered in §§6–10. This is followed by a section (§11) on multivalent functions and three sections (§§12–14) on functions of bounded index. An entire function $f(z)$ is said to be of bounded index if there exists an integer N, independent of z, such that

$$\max_{0 \leqq s \leqq N} \left\{ \frac{|f^{(s)}(z)|}{s!} \right\} \geqq \frac{|f^{(j)}(z)|}{j!}, \tag{1.1}$$

for $j = 1, 2, \ldots$ and for all z. The smallest such integer N is called the index of f. An entire function f of bounded index N is of exponential type not exceeding $(N + 1)$. Finally we mention some unsolved problems.

2. **Conditions for the univalence of f.** Let

$$f(z) = \sum_0^\infty a_n z^n, \qquad |z| < 1. \tag{2.1}$$

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[1] In this article we shall not consider meromorphic univalent functions.

If $a_1 \neq 0$ and

$$\sum_{2}^{\infty} n|a_n| \leqq |a_1|, \tag{2.2}$$

then f is analytic and univalent in D and continuous on the closure of D. To prove this, let $z_1, z_2 \in D$, $\max_{i=1,2}|z_i| = r < 1$, $z_1 \neq z_2$. Then

$$\left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| = \left| a_1 + \sum_{2}^{\infty} a_n(z_2^{n-1} + z_2^{n-1}z_1 + \cdots + z_1^{n-1})\right|$$

$$\geqq |a_1| - \sum_{2}^{\infty} n|a_n|r^{n-1} > 0.$$

This implies that f is univalent in D. Further, for every $N \geqq 1$,

$$\sum_{0}^{N} |a_n| \leqq |a_0| + |a_1| + \sum_{2}^{\infty} \frac{n|a_n|}{n} \leqq |a_0| + \frac{3|a_1|}{2},$$

and continuity of f follows.

If the radius of convergence of the series in (2.1) defining f is R, then f is univalent in $|z| < \rho \leqq R$, if $a_1 \neq 0$ and

$$\sum_{n=2}^{\infty} n|a_n|\rho^{n-1} \leqq |a_1|. \tag{2.3}$$

Let f be analytic in D. If f is univalent in D then $f'(z) \neq 0$ in D [**32**, p. 23]. If

$$\mathrm{Re}(af'(z)) > 0, \qquad z \in D, \tag{2.4}$$

for some complex number a, $|a| = 1$, then f is univalent in D. This follows immediately from the following integral expression

$$\mathrm{Re}\left\{\frac{a(f(z_2) - f(z_1))}{z_2 - z_1}\right\} = \int_0^1 \mathrm{Re}\{af'((1 - w)z_1 + wz_2)\}\, dw.$$

Another criterion for univalence of f ([**62**]; see also [**29**]) is as follows. Let

$$\{w, z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

be the Schwarzian derivative of $w = f(z)$ with respect to z. In order that $w = f(z)$ be univalent in D it is necessary that

$$|(w, z)| \leqq 6/(1 - |z|^2)^2$$

and sufficient that

$$|(w, z)| \leqq 2/(1 - |z|^2)^2.$$

Becker [**2**] has recently proved that f is univalent in D if

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{1}{(1-|z|^2)}.$$

3. **Class** S. Let S denote the collection of functions f analytic and univalent in D and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Thus $f \in S$ can be written as

$$(3.1) \qquad f(z) = z + \sum_{2}^{\infty} a_n z^n, \qquad |z| < 1.$$

Bieberbach [**6**] proved in 1916 that, for $f \in S$,

$$(3.2) \qquad |a_2| \leqq 2$$

with equality if and only if

$$(3.3) \qquad f(z) = K_\alpha(z) \equiv z/(1 - ze^{i\alpha})^2 \qquad (\alpha \text{ real}).$$

This function K_α (Koebe function) maps D on the whole plane slit radially from $w = -\frac{1}{4}e^{-i\alpha}$ to infinity. It is extremal not only for a_2 but also for a number of other problems. Since $|a_n| = n$, $n = 2, 3, \ldots$ for this function K_α, it was conjectured that, for $f \in S$,

$$(3.4) \qquad |a_n| \leqq n, \qquad n = 2, 3, \ldots,$$

with equality only for the Koebe function. This conjecture, called the Bieberbach conjecture, was proved for $n = 3$ by Loewner [**58**] in 1923, for $n = 4$ by Charzynski and Schiffer ([**18**]; see also [**30**]) in 1960, and for $n = 6$ by Pederson [**66**] in 1968 and Ozawa [**64**] in 1969 independently of each other. Garabedian and Schiffer [**31**] proved that (3.4) holds for a function $f \in S$ which is "close enough" to the Koebe function and Aharonov has shown (3.4) to hold if $|a_2| < 0.867$ ([**1**]; see also [**9**]).

For each fixed $f \in S$, Hayman (see [**38**, pp. 112–113]) has shown that $|a_n| \leqq n$ ($n > n_0(f)$). For all $n \geqq 2$, Littlewood proved in 1925 (see [**38**, p. 10]) that $|a_n| < en$. This estimate has recently been improved to $|a_n| < 1.081n$ ($n \geqq 2$) by Carl H. Fitzgerald (see also [**32**, p. 612]).

4. **Subclasses of** S. A function $f \in S$ is said to be starlike univalent in D, or briefly starlike in D if $f(D)$ is starlike with respect to the origin $w = 0$. A necessary and sufficient condition for $f \in S$ to be starlike in D is that [**63**, pp. 220–222], [**38**, pp. 14–16],

$$(4.1) \qquad \operatorname{Re}(zf'(z)/f(z)) > 0, \qquad |z| < 1.$$

We shall denote this subclass of functions by S^*. From (4.1) it is easy to obtain, for $f \in S^*$, the following integral representation formula

(4.2) $$z\frac{f'(z)}{f(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}\,dV(t)$$

where $V(t)$ is an increasing function of t, $V(t) - t$ has period 2π and $(1/2\pi)\int_0^{2\pi} dV(t) = 1$. A second subclass of S is the class of convex univalent functions. We say that $f \in S$ is convex univalent in D if $f(D)$ is a convex set. We denote this subclass of S by K. A necessary and sufficient condition for $f \in S$ to be in K is that [**38**, pp. 140–141], [**32**, p. 166],

(4.3) $$\operatorname{Re}(1 + zf''(z)/f'(z)) > 0, \qquad |z| < 1.$$

If $f \in K$ then $|a_n| \leqq 1$. If $f \in S^*$ then $|a_n| \leqq n$.

A third subclass of functions is the class of close-to-convex functions introduced by Kaplan [**46**]. A function $f \in S$ is close-to-convex if and only if

(4.4) $$\operatorname{Re}(f'(z)/\phi'(z)) > 0, \qquad |z| < 1,$$

where $\phi(z)/\phi'(0) \in K$. (If f is analytic in D and satisfies the close-to-convex condition (4.4) then it is univalent.) For this class (3.4) also holds. If f is defined by (2.1) and satisfies (2.2) and if $f(0) = 0, f'(0) \neq 0$, then f is starlike in D [**33**]. From this we can conclude that if [**33**]

(4.5) $$\sum_{k=2}^{\infty} k^2|a_k| \leqq |a_1|$$

then f is convex in D.

For more information on various problems of univalent function theory we refer the reader to five excellent survey articles by Bernardi [**3**], Hayman [**39**], Goluzin [**32**, pp. 577–628], Goodman [**35**] and Robertson [**73**]. We list some recent papers in the bibliography at the end and refer to an exhaustive bibliography by Bernardi [**4**], for books and periodical literature up to 1965.

5. **Functions with univalent derivatives.** Let $f \in S$ and let E denote the subclass

(5.1) $$E = \{f \mid f \in S, f^{(k)} \text{ is univalent in } D \text{ for } k = 1, 2, \ldots\}.$$

If $f \in E$ then f must be a transcendental entire function of exponential type, that is,

(5.2) $$\limsup_{r\to\infty}\frac{\log M(r,f)}{r} \equiv T^* < \infty,$$

where as usual $M(r,f) = \max_{|z|=r}|f(z)|$. (Note that functions, for which $0 \leqq T^* < \infty$, and in particular functions of order less than one, are all functions of exponential type.) More precisely we have [**84**]

$$(5.3) \qquad |f(z)| \leqq \frac{\exp(2\alpha|z|) - 1}{2\alpha},$$

where $\alpha = \sup\{|a_2| : f \in E\}$ and

$$(5.4) \qquad \pi/2 \leqq \alpha < 1.7208.$$

To prove this we note that if $f \in E$ then $a_{n+1} \neq 0$. Define F_n in D by

$$F_n(z) = \frac{f^{(n)}(z) - n!a_n}{(n+1)!a_{n+1}}.$$

Then $F_n \in E$ and we have

$$|a_{n+2}| \leqq \frac{2\alpha|a_{n+1}|}{n+2}.$$

An inductive argument gives $|a_n| \leqq (2\alpha)^{n-1}/n!$ $(n \geqq 2)$. This implies that f is entire and satisfies (5.3). Since $|a_2^2 - a_3| \leqq 1 - \{M(1)\}^{-2}$ ([**44**], [**88**]), we have

$$\alpha^2 \leqq 3(1 - 4\alpha^2/(e^{2\alpha} - 1)^2).$$

This implies the right-hand inequality in (5.4). To complete the proof of (5.4) we observe that $\phi(z) = (\exp(\pi z) - 1)/\pi \in E$ and a_2 for this function is $\pi/2$.

We note here that the property of univalence is only one of the properties which forces f to be entire. Consider a property (A) which a function analytic in D is able to possess. We say that (A) is an admissible property provided the following hold: (i) if f has (A) then $f'(0) \neq 0$. (ii) If f has (A) and if b and c are complex numbers with $b \neq 0$, then the function $F(z) = bf(z) + c$ also has (A). Let T be the family of functions f, analytic in D, of the form (3.1). Let $T(A)$ be the subclass of T such that if $f \in T(A)$ then $f^{(n)}$ has property (A) for $n = 0, 1, 2, \cdots$. Suppose that $T(A)$ is not empty and let $\alpha_A = \sup\{|a_2| : f \in T(A)\}$. If $\alpha_A < \infty$ and $f \in T(A)$ then f is a transcendental entire function of exponential type not greater than $2\alpha_A$ [**86**]. For instance one can take property (A) to be property (K). We say that f has (K) if f is convex univalent in D. Then (K) is an admissible property. Further $\alpha_K = \sup\{|a_2| : f \in T(K)\}$ lies between $\frac{1}{2}$ and 0.6838 [**86**].

6. **Not all derivatives univalent.** Let f be defined in D by (2.1) and let $\{n_k\}_1^\infty$ be a sequence of strictly increasing positive integers. Suppose that each $f^{(n_k)}$ is univalent in D. Let R be the radius of convergence of the series in (2.1). If the sequence $\{n_k\}$ does not increase very rapidly, we may have $R > 1$. Thus, for instance [**86**],

$$\liminf_{k\to\infty}(n_1\cdots n_k)^{1/n_k} \leqq R\limsup_{k\to\infty} 4^{k/n_k} \leqq 4R. \tag{6.1}$$

From (6.1) it is easy to show that if $n_{k+1} - n_k = o(\log k)$ then $R = \infty$ and f is entire. If $n_{k+1} - n_k = O(1)$ then f is of exponential type.

A more general result of this type is as follows. Let $\phi(x)$ and $\theta(x)$ be two slowly oscillating functions (see [**86**] and the references given there) and let $1 \leqq \phi(k) \leqq n_k - n_{k-1} \leqq \theta(k)$ for $k = 2, 3, \cdots$. If each $f^{(n_k)}$ is univalent in D and

$$\limsup_{k\to\infty}\frac{\theta(k)\log\theta(k)}{\phi(k)\log k} = \alpha < 1,$$

then f is an entire function of order not greater than $1/(1-\alpha)$.

If however the sequence $\{n_k\}$ increases very rapidly, say

$$n_{k+1} \geqq n_k \log n_k \log\log n_k,$$

then R may not exceed unity. In fact there exists [**86**] a function f, analytic in D and an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that f and each $f^{(n_k)}$ map D univalently onto convex domains and yet the unit circle is the natural boundary of f.

7. **Derivatives with varying radii of univalence.** Let $\rho(f)$ be the largest number with the property that f is analytic and univalent in an open disc about the origin of radius ρ. We shall write $\rho(f^{(n)}) = \rho_n$. Suppose now that f is defined by $f(z) = \sum_0^\infty a_n z^n$. Let R denote the radius of convergence of this series. Then we have [**85**]

$$\liminf_{n\to\infty} n\rho_n \leqq 4R, \tag{7.1}$$

and

$$R\log 2 \leqq \limsup_{n\to\infty} n\rho_n. \tag{7.2}$$

If $|a_{n-1}/a_n|$ is ultimately a nondecreasing sequence, then

$$R\log 2 \leqq \liminf_{n\to\infty} n\rho_n \leqq \limsup_{n\to\infty} n\rho_n \leqq 4R. \tag{7.3}$$

Thus (a) if f is a transcendental entire function then $\limsup_{n\to\infty} n\rho_n = \infty$, and (b) if $\lim_{n\to\infty} n\rho_n = \infty$, then f is a transcendental entire function. (See also [**85**, Theorem 3].) The converse of (a) is false. There exists a function f analytic in the unit disc and in no larger disc $|z| < R$, where $R > 1$, such that $\limsup n\rho_n = \infty$. The converse of (b) is also false [**85**].

8. **Radii of univalence and entire functions.** Let f be a transcendental entire function of order Λ and lower order λ (see [**8**, p. 8]). When $0 < \Lambda < \infty$, let $T = \limsup_{r\to\infty} \log M(r)/r^{\Lambda}$ denote the type and $t =$

$\liminf_{r\to\infty} \log M(r)/r^{\Lambda}$ denote the lower type. The following theorems are due to Boas, Pólya and Takenaka respectively.

THEOREM A [7]. *If $f(z)$ is a transcendental entire function and if*

$$T^* = \limsup_{r\to\infty} \frac{\log M(r)}{r} < \log 2, \tag{8.1}$$

then there is a sequence $\{n_p\}_{p=1}^{\infty}$ such that $\rho_{n_p} = \rho(n_p) \geqq 1$ for all p.

Levinson [56] supplied a second proof of this. Boas also pointed out that, if $T^* = 0$, then

$$\limsup_{n\to\infty} \rho_n = \infty. \tag{8.2}$$

THEOREM B [67]. *If $f(z)$ is a transcendental entire function of order Λ, then*

$$\liminf_{n\to\infty} \frac{\log \rho_n}{\log n} \leqq \frac{1-\Lambda}{\Lambda} \leqq \limsup_{n\to\infty} \frac{\log \rho_n}{\log n}. \tag{8.3}$$

THEOREM C [92]. *If $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of complex numbers of modulus not exceeding one and if $f(z)$ is an entire function of exponential type less than* $\log 2$, *then $f(z)$ vanishes identically if $f^{(n)}(\alpha_n) = 0$, $n = 0, 1, 2, \cdots$.*

We give improved versions of these theorems. Let us denote by $\nu(r)$ $(0 < r < +\infty)$ the central index of the series $f(z) = \sum_0^{\infty} a_n z^n$ for $|z| = r$. Then

$$|a_n| r^n \leqq |a_{\nu(r)}| r^{\nu(r)}, \qquad n = 0, 1, 2, \cdots.$$

Let

$$\limsup_{r\to\infty} \frac{\nu(r)}{r} = \gamma, \qquad \liminf_{r\to\infty} \frac{\nu(r)}{r} = \delta. \tag{8.4}$$

Then we have [85]

$$\liminf_{n\to\infty} \frac{\log \max(1, n\rho_n)}{\log n} \leqq \frac{1}{\Lambda}, \tag{8.5}$$

$$\frac{1-\lambda}{\lambda} \leqq \limsup_{n\to\infty} \frac{\log \rho_n}{\log n}, \tag{8.6}$$

$$\frac{\log 2}{\delta} \leqq \limsup_{n\to\infty} \rho_n \tag{8.7}$$

and

$$\liminf_{n\to\infty} n^{\Lambda-1}\rho_n^{\Lambda} \leqq \frac{4^{\Lambda}}{\Lambda T}. \tag{8.8}$$

Hence if $\Lambda > 1$, $\liminf_{n\to\infty} \rho_n = 0$ and if $\Lambda = 1$, then since $\delta \leqq t^*$ $(=\liminf_{r\to\infty} \log M(r)/r) \leqq T^*$,

$$\frac{\log 2}{t^*} \leqq \limsup_{n\to\infty} \rho_n; \qquad \liminf_{n\to\infty} \rho_n \leqq \frac{4}{T^*}. \tag{8.9}$$

The inequalities (8.5)–(8.6) imply Theorem B and (8.7) implies Theorem A. Theorem C follows immediately from (8.7) since $\rho(f^{(n)}) \leqq r_{n+1}^*$ where r_k^* denotes the absolute value of the zero z_k^* of $f^{(k)}$ which is nearest to the origin. (If $f^{(k)}$ has no zero then $r_k^* = \infty$.)

For entire functions defined by gap power series, (8.6) and (8.7) give, in general, better results than Theorems A–C. Let

$$f(z) = \sum a_{n_k} z^{n_k} \qquad (a_{n_k} \neq 0, k = 1, 2, \ldots), \tag{8.10}$$

be a transcendental entire function and suppose that

$$\liminf_{k\to\infty} \log n_k/\log n_{k+1} = \chi < 1. \tag{8.11}$$

Then $\lambda \leqq \Lambda\chi$ [**93**] and (8.5) and (8.6) give more information than Theorem B. If we suppose now that $\Lambda \geqq 1$ but $\Lambda\chi < 1$ then $\lambda < 1$, $\delta = 0$ and (8.7) implies that $\limsup_{n\to\infty} \rho_n = \infty$. Thus Theorems A and C hold for every function f, of any finite order Λ and of the form (8.10) with gaps satisfying the condition (8.11) and $\Lambda\chi < 1$.

If $f(z) = \sum_0^\infty a_n z^n$ and $|a_n/a_{n+1}|$ is ultimately a nondecreasing function of n, tending to ∞, then f is entire and [**85**]

$$\frac{\log 2}{\gamma} \leqq \liminf_{n\to\infty} \rho_n \leqq \frac{4}{\gamma}, \tag{8.12}$$

$$\frac{\log 2}{\delta} \leqq \limsup_{n\to\infty} \rho_n \leqq \frac{4}{\delta}. \tag{8.13}$$

9. **Whittaker constant.** Consider again Theorem A and let α be the least upper bound of all numbers which can replace log 2 in Theorem A. Read [**71**] has shown that $\alpha \geqq 0.7259$. Let W be the least upper bound of numbers which can replace log 2 in Theorem C. This number is called the Whittaker constant. It is known that (see [**71**], [**11**] and the references given there) $0.7259 \leqq W < 0.7378$ but the exact value is unknown. Recently Buckholtz [**11**] has shown that $\alpha = W$.

A simple example of a function f of order one such that each of f, f', $f'', \ldots$ has a zero in the closed disc $|z| \leqq 1$ is $f(z) = \sin(\pi z/4) - \cos(\pi z/4)$.

There exist extremal functions for this problem. In fact Evgrafov (see [**11**]) has shown that there is an entire function f of exponential type W such that each of $f, f', f'', \ldots$ has a zero in the disc $|z| \leqq 1$.

Mention must be made here of a related result due to Erdös and Renyi [**26**]. Let f be entire and denote by $x = H(y)$ the inverse function of $y = \log M(x)$. Then

$$\liminf_{k \to \infty} \frac{H(k)}{kr_k^*} \leqq \frac{e}{\log 2}.$$

10. **Functions in** E. (i) Consider first a function f defined by the power series (2.1) and suppose that $a_n \neq 0$, $n|(a_n/a_{n-1})| \leqq \log 2$ for $n = 2, 3, \cdots$. Then f is entire and it can be shown that $(f(z) - a_0)/a_1 \in E$.

(ii) We now consider functions with all zeros on a ray. Let Ω denote the family of transcendental entire functions f of the form

$$f(z) = ze^{\beta z} \prod_1^N (1 - z/z_k) \tag{10.1}$$

where $0 \leqq N \leqq \infty$ (if $N = 0$, the product disappears) and (a) all z_k have the same argument, (b) $\beta z_1 \leqq 0$ and (c) $1 < |z_1| \leqq |z_2| \leqq \cdots$. If $f \in \Omega$ and is univalent in D then [**87**]

$$|\beta| + \sum_{k=1}^N \frac{1}{|z_k| - 1} \leqq 1. \tag{10.2}$$

In fact (10.2) holds if and only if f is starlike in D and all its derivatives are close-to-convex there. Further, if $\{z_k^{(1)}\}_{k=0}^N$ are the zeros of f', then f and all its derivatives are univalent in D and map D onto convex domains if and only if [**87**]

$$|\beta| + \sum_{k=0}^N \frac{1}{|z_k^{(1)}| - 1} \leqq 1. \tag{10.3}$$

This result implies that $E \cap \Omega = S \cap \Omega$ and that $f \in E \cap \Omega$ if and only if (10.2) holds.

(For the univalence of an entire function of any order see [**61**].)

(iii) If all zeros of f do not lie on a ray then some derivative $f', f'', \ldots$ may have zeros in the unit disc (e.g., $f(z) = \sin(\pi z/2)/(\pi/2)$) and then f will not belong to E. If however f is of genus zero, and $f(0) = 0, f'(0) = 1$, and the zeros are widely spaced, then $f \in E$. We shall say that a function f has "fourly-spaced" zeros if

$$|z_1| \geqq 4, \qquad |z_{k+1}| \geqq 4^k |z_k|, \qquad k \geqq 1. \tag{10.4}$$

Let

$$P(z) = \prod_1^\infty (1 - z/z_k), \qquad f(z) = zP(z). \tag{10.5}$$

Then [**78**], $f \in E$. It is possible to improve the constant 4.

11. **Multivalent functions.** A function f is said to be p-valent in D if it is analytic in D, if the equation

$$f(z) = w \tag{11.1}$$

has p distinct roots in D for some particular w, and if for each complex w, equation (11.1) does not have more than p roots in D. The function f is also said to have valence p in D. When $p = 1, f$ is univalent in D.

Goodman [**34**] considered the sum $(f + g)/2$ and the product $(fg)^{1/2}$ when f and g both belong to S and showed that there exist two pairs of functions f_1, g_1 and f_2, g_2 each function belonging to S such that the sum $(f_1 + g_1)/2$ and, the product $(f_2(z)g_2(z))^{1/2} = z + \cdots$, both have valence ∞ in D.

We now define areally mean p-valent (a.m.p.v.) functions. Let p be a positive number and denote by $n(w)$ the number of roots of the equation (11.1) in D. If f is analytic in D and, for every positive R,

$$\frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R n(\rho e^{i\phi})\rho \, d\rho \, d\phi \leqq p, \tag{11.2}$$

then f is said to be a.m.p.v. in D. A condition for f to be a.m.p.v. is as follows. Let

$$\sum_1^\infty |a_n| = \bar{S} < |a_0|, \qquad \sum_1^\infty n|a_n|^2 = A < \infty. \tag{11.3}$$

Then $f(z) = \sum_0^\infty a_n z^n$ is a.m.p.v. in D for all large p such that ([**39**], [**68**])

$$|a_0| > (A/p)^{1/2} + \bar{S}. \tag{11.4}$$

If f is a.m.p.v. in D and is normalized and $p = 1$, then $|a_2| \leqq 2$ [**89**]. A bound on $|f|$ is given by the following theorem due to Cartwright, Spencer and Hayman.

THEOREM [**38**, p. 31]. *Suppose that* $f(z) = \sum_0^\infty a_n z^n$ *is a.m.p.v. in D. Then*

$$M(r,f) < A(p)\mu_p(1 - r)^{-2p} \qquad (0 < r < 1),$$

where $\mu_p = \max_{0 \leqq \nu \leqq p} |a_\nu|$ *and* $A(p) \leqq (p + 2)2^{3p-1} \exp(p\pi^2 + \frac{1}{2})$.

This upper bound on the constant $A(p)$ is due to Jenkins and Oikawa

[**45**]. In §5(i) we have seen that if $f \in S$ and each $f^{(k)}$ $(k = 1, 2, \dots)$ is univalent in D then f is a transcendental entire function of exponential type. This result holds under a less restrictive hypothesis. Suppose f is not a polynomial and each $f^{(k)}$ $(k = 0, 1, \dots)$ is a.m.p.v. in D. Then [**81**], f is an entire function of exponential type not exceeding $A(p)e(P + 2)^{2P}(P + 1)$ where $P = [p]$ is integer part of p. If each $f^{(n_j)}$, $j = 1, 2, \dots,$ is a.m.p.v. in D and

$$(11.5) \qquad \lim_{j \to \infty}(n_{j+1} - n_j) = \infty, \qquad n_j = O\left(\sum_{k=1}^{j} \log n_k\right),$$

then also f must be entire.

12. **Entire functions of bounded index.** Let $f(z) = \sum_{n=0}^{\infty} A_n(z - a)^n$ be an entire function. Since the coefficients tend to zero, there exists a smallest integer $N_a \geqq 0$ such that $|A_{N_a}| \geqq |A_n|$ for all n. If the integers N_a are all bounded above then f is said to be of bounded index and the smallest integer N, such that for all numbers a, $N_a \leqq N$, is called the index of f (cf., [**55**], [**36**]). This is equivalent to the definition given in §1. As we pointed out a function of bounded index N is of exponential type not exceeding $N + 1$. This result is sharp [**76**]. Denote the class of all functions of bounded index by B. The functions e^z, $\sin z$, $\cos z$ are all in B.

The Bessel function $J_k(z)$ of integer order k is of index N such that $k \leqq N \leqq 2k - 1$ ([**52**]; see also [**60**]). Any entire function f satisfying a linear differential equation [**77**]

$$(12.1) \qquad P_0(z)\frac{d^n f}{dz^n} + P_1(z)\frac{d^{n-1} f}{dz^{n-1}} + \cdots + P_n(z) f = Q(z),$$

where P_j $(j = 0, 1, \dots, n)$ and Q are polynomials and $\deg P_j \leqq \deg P_0$ is in class B.

Functions with zeros of arbitrarily large multiplicity are obviously of unbounded index. But there are functions [**79**] of unbounded index and having simple zeros.

The asymptotic properties of $\log M(r, f)$ do not help to prove the boundedness (or the unboundedness) of the index, except that if $T^* = \infty$ then $f \in CB$ (the class of entire functions of unbounded index). In fact if F is any transcendental entire function then there are two entire functions $g \in CB$ [**70**] and $f \in CE$ (the class of entire functions not belonging to E) such that

$$\log M(r, g) \sim \log M(r, F) \sim \log M(r, f).$$

For f we simply take $f(z) = F(z) - F''(0)z^2/2!$.

We mentioned in §11 that there exist functions f and g in S such that $(f + g)/2$ is not in S. Pugh [**69**] showed that the sum of two functions each in B, need not be in B.

The class B is not closed under differentiation. There exists [**80**] an entire function F in B such that the derivative F' is in CB. If the derivative f' is of bounded index $N_{f'}$, f is also of bounded index N_f and $N_f \leqq N_{f'} + 1$ [**80**].

The functions P and f defined by (10.4) and (10.5) are both in B. (Cf. [**70**]. The constant 5 in [**70**] has been improved to 4 by Mrs. Amy King in her Ph.D dissertation.) In fact, we have, for all z,

$$\max\{|P(z)|, |P'(z)|\} \geqq |P^{(n)}(z)|, \qquad n = 2, 3, \cdots.$$

Furthermore each $P^{(k)}$, $k = 0, 1, 2, \ldots$, is of index 1.

Consider now functions with real zeros a_n. Suppose $a_1 > 0$, $a_{n+1} - a_n \geqq b_n$ $(n \geqq 1)$ where the sequence $\{b_n\}_1^\infty$ is positive and nondecreasing and $\sum_1^\infty 1/nb_n < \infty$. Then [**82**],

$$f(z) = e^{\alpha z + \beta} \prod_1^\infty (1 - z/a_n), \tag{12.2}$$

where α and β are any complex numbers, is in B. If in (12.2) we assume that $a_1 > 0$, $a_{n+1}/a_n \geqq \gamma > 1$, then each $f^{(k)}$, $k = 0, 1, \ldots$, is in B [**54**].

We can consider entire functions f satisfying conditions similar to (1.1) and obtain the conclusion that f must be of exponential type [**37**], [**83**].

(a) Let $p \geqq 1$ and

$$I(l, r) = \left\{ \int_0^{2\pi} |f^{(l)}(re^{i\theta})|^p \, d\theta \right\}^{1/p}.$$

Let c be a positive constant. Suppose that there exists a positive integer N (independent of z) such that for $k = 0, 1, 2, \ldots, N$, the following inequality

$$\sum_{j=0}^{N} \frac{I(k + j, r)}{j!} \geqq c \sum_{j=N+1}^{\infty} \frac{I(k + j, r)}{j!}$$

holds for all z with $|z| = r$ sufficiently large. Then f is of exponential type and

$$T^* \leqq 1 + 2\log(1 + 1/c) + \log(2N)!.$$

(b) Let c be a positive constant. Suppose that there exist two non-negative integers k and N (independent of z) such that f satisfies one of the following, for all z with $|z|$ sufficiently large:

(i) $$\sum_{j=0}^{N} \frac{|f^{(k+j)}(z)|}{j!} \geqq c \sum_{j=N+1}^{\infty} \frac{|f^{(k+j)}(z)|}{j!},$$

(ii) $$\sum_{j=0}^{N} \frac{M(r, f^{(k+j)})}{j!} \geqq c \sum_{j=N+1}^{\infty} \frac{M(r, f^{(k+j)})}{j!},$$

then f is of exponential type and

$$T^* \leqq \max\left\{N, \min_{1 \leqq j \leqq N}\left(\frac{(N+j)!(N+1)}{(N!)c}\right)^{1/j}, \left(\frac{(2N+1)!}{(N!)c}\right)^{1/(N+1)}\right\}.$$

13. **The space of entire functions.** Following Iyer [**43**] we define a metric on the space of all entire functions Γ. (This space includes all polynomials and constant zero.) Let $f(z) = \sum_0^\infty a_n z^n$ and $g(z) = \sum_0^\infty b_n z^n \in \Gamma$ and define

$$d(f, g) = \sup\{|a_0 - b_0|, |a_n - b_n|^{1/n}: n = 1, 2, \ldots\}.$$

Then d is a metric and (Γ, d) is a complete metric space [**43**]. Let

$$B_n = \{f \in (\Gamma, d) | f \text{ is of index not exceeding } n\}.$$

We consider $B = \bigcup_{n=0}^\infty B_n$ as a subspace of (Γ, d). It can be shown that [**25**] B_n is nowhere dense in B and thus B is of the first category.

14. **Some applications to summability methods.** Let f be entire and $\{z_i\}_{i=0}^\infty$ a sequence of complex numbers. We define the matrix transformation $A(f, z_i) = (a_{n,k})$ by

$$f(z) = \sum_{k=0}^{\infty} a_{n,k}(z - z_n)^k \quad \text{for } n = 0, 1, \cdots.$$

We now state some recent results of Fricke and Powell.

I [**28**]. If $f \in B$ then $A(f, z_i) = (a_{n,k})$ is not regular for any sequence $\{z_i\}_{i=0}^\infty$. (A transformation $A = (a_{n,k})$ is regular if it transforms every convergent sequence into a sequence converging to the same limit. See [**41**, p. 43].)

Define a sequence $\{a_n\}_0^\infty$ to be entire if $f(z) = \sum_0^\infty a_n z^n$ is an entire function. An entire sequence $\{a_n\}_0^\infty$ is said to be a sequence of bounded index if $f(z) = \sum_0^\infty a_n z^n \in B$. We denote by ε the set of all entire sequences and by $\mathscr{B}$ the set of all entire sequences of bounded index. An infinite matrix $A = (a_{n,k})$ of complex numbers which transforms ε into ε is said to be an ε-ε method (entire method).

II [**27**]. A matrix $A = (a_{n,k})$ is an ε-ε method if and only if for each integer $q > 0$, there exists an integer $p > 0$ and a constant $M > 0$ such that

$$|a_{n,k}|q^n \leqq Mp^k \quad \text{for all } n, k = 0, 1, \cdots.$$

Let $A'(f, z_i) = (b_{n,k})$ denote the transpose of $A(f, z_i) = (a_{n,k})$, that is, $b_{n,k} = a_{k,n}$.

III [**28**]. If $f \in B$ then for any sequence $\{z_i\}_{i=0}^\infty$, $A'(f, z_i) = (b_{n,k})$ is an ε-ε method if and only if for each integer $n > 0$ there exist an integer $p > 0$ and a constant $M > 0$ such that

$$|f^{(n)}(z_k)| \leqq p^k M \quad \text{for } k = 0, 1, \cdots.$$

The condition that $f \in B$ is essential in III.

We now define the l-l method. Let s be the set of all sequences of complex numbers. Let

$$l = \left\{ x = \{x_n\}_{n=0}^{\infty} \in s \,\middle|\, \sum_0^{\infty} |x_n| < \infty \right\}.$$

A matrix $A = (a_{n,k})$ that maps l into itself is said to be an l-l method. Knopp and Lorentz [**49**] proved that a matrix $A = (a_{n,k})$ is an l-l method if and only if there exists a constant $M > 0$ such that

$$\sum_{n=0}^{\infty} |a_{n,k}| \leqq M \quad \text{for } k = 0, 1, \cdots.$$

IV [**28**]. Let $f \in B$ and $\{z_i\}_{i=0}^{\infty}$ be a sequence of complex numbers. If either $A(f, z_i) = (a_{n,k})$ or $A'(f, z_i) = (b_{n,k})$ is an l-l method then $A'(f, z_i)$ is an ε-ε method.

Finally we give a matrix which transforms $\mathscr{B}$ into $\mathscr{B}$.

Let the Taylor matrix $T(\xi) = (a_{n,k})$ be defined by

$$a_{n,k} = \binom{k}{n}(1 - \xi)^{n+1}\xi^{k-n}, \quad \text{if } k \geqq n,$$
$$= 0, \qquad \text{otherwise},$$

where ξ is a complex number.

V [**28**]. The Taylor matrix $T(\xi) = (a_{n,k})$ transforms $\mathscr{B}$ into $\mathscr{B}$ for any complex number ξ.

15. **Conjectures and open problems.** We now list some problems and conjectures connected with two classes E and B.

CONJECTURE 1. If ϕ is any transcendental entire function such that

$$\limsup_{r \to \infty} \frac{\log M(r, \phi)}{r} \leqq \pi,$$

there exists an entire function $f \in E$ such that $\log M(r, \phi) \sim \log M(r, f)$ $(r \to \infty)$.

CONJECTURE 2. If ϕ is any entire function of exponential type, there exists an entire function $f \in B$ such that $\log M(r, \phi) \sim \log M(r, f)$ $(r \to \infty)$.

For some theorems of this type, but not connected with E or B, see [**22**], [**19**], [**20**].

CONJECTURE 3. $W = 2/e$.

CONJECTURE 4. If $\sum_{p=1}^{\infty} 1/n_p = \infty$ and $\rho(f^{(n_p)}) \geqq 1$ for $p = 1, 2, \ldots,$ then f is entire.

In the following problems 1–4, $f \in E$.

1. What is the smallest zero that f can have? (Exclude $z = 0$.)
2. What is the largest circle center origin covered by $f(D)$?
3. Find bounds on $|a_2^2 - a_3|$.
4. Find $\alpha = \sup\{|a_2| \mid f \in E\}$.
5. Find $\alpha_K = \sup\{|a_2| \mid f \in T(K)\}$.
6. Let f be entire and satisfy a differential equation of the form (12.1). Assume P_j $(j = 0, 1, \ldots, n)$ and Q are polynomials and $\deg P_j \leqq \deg P_0$. Then f is of bounded index. Find an estimate for the index.

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