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EXTENDING FOURIER TRANSFORMS INTO SZ.-NAGY-FOIAŞ SPACES[1]

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1. **Introduction.** Let $s(z)$ be a function in the unit ball of H^∞ of the unit disc. Let $\Delta(e^{it}) = (1 - |s(e^{it})|^2)^{1/2}$ and let $E = \{t | \Delta(e^{it}) > 0\}$. We consider the subspace $\mathscr{M}_s$ of $H^2 \oplus L^2(E)$ of pairs of the form $(s(z)f(z), \Delta(e^{it})f(e^{it}))$ for $f \in H^2$. The *Sz.-Nagy-Foiaş space associated with s* is the orthogonal complement $\mathscr{M}_s^\perp = [H^2 \oplus L^2(E)] \ominus \mathscr{M}_s$. The function s is *inner* precisely when E is a zero set, and in that case $\mathscr{M}_s$ reduces to the invariant subspace sH^2.

In the latter case, two "Fourier transforms" have recently been defined, from various L^2 spaces into $\mathscr{M}_s^\perp = H^2 \ominus sH^2$. The first such unitary operator, defined by Ahern and Clark [**1**] and Kriete [**3**] is obtained as follows. Let σ be a singular measure without atoms on $[0, 2\pi]$ and set

$$(1) \qquad s_\lambda(z) = \exp\left[-\int_0^\lambda (e^{i\theta} + z)/(e^{i\theta} - z)\, d\sigma(\theta)\right], \qquad s(z) \equiv s_{2\pi}(z).$$

An operator $\mathscr{U}$ from $L^2(d\sigma)$ to $\mathscr{M}_s^\perp$ is defined by

$$(\mathrm{A}) \qquad (\mathscr{U}f)(z) = 2^{1/2} \int_0^{2\pi} f(\lambda) s_\lambda(z)(1 - e^{-i\lambda} z)^{-1}\, d\sigma(\lambda).$$

Then $\mathscr{U}$ is unitary and satisfies

$$(2) \qquad \mathscr{U}^* T \mathscr{U} = (I - K)M,$$

where T is the restricted shift on $\mathscr{M}_s^\perp$:

$$Tg = P_{\mathscr{M}_s^\perp} zg$$

and, for $f \in L^2(d\sigma)$,

$$(3) \qquad Mf = e^{it} f(t), \qquad Kf = 2\int_0^t e^{-\sigma([\lambda, t])} f(\lambda)\, d\sigma(\lambda).$$

The second transform was defined by the author in [**2**]. Let ν be an arbitrary singular measure on $[0, 2\pi]$ and let

$$(4) \qquad s(z) = \left[\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\nu(\theta) - 1\right]\left[\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\nu(\theta) + 1\right]^{-1}.$$

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Define $\mathscr{U}$ from $L^2(d\nu)$ to $\mathscr{M}_s^\perp$ by

(B) $$(\mathscr{U}f)(z) = (1 - s(z)) \int_0^{2\pi} f(\lambda)(1 - e^{-i\lambda}z)^{-1}\, d\nu(\lambda).$$

Then $\mathscr{U}$ is unitary and satisfies

(5) $$\mathscr{U}^* T \mathscr{U} f = Hf$$

where

(6) $$(Hf)(t) = e^{it}[I - (1 + s(0))P_{\{e^{-it}\}}]f.$$

The transforms have been generalized to map into more general Sz.-Nagy-Foiaş spaces. Kriete [3], working under the assumption that the Cauchy kernels

$$K_\zeta(z, t) = ((1 - \bar{s}(\zeta)s(z))/(1 - \bar{\zeta}z), -\bar{s}(\zeta)\Delta(e^{it})/(1 - \bar{\zeta}e^{it}))$$

($|\zeta| < 1$) span $\mathscr{M}_s^\perp$, extended transform (A) by designating which function in $L^2(d\sigma)$ be sent into K_ζ. The author [2] obtained a similar extension of transform (B), again under the assumption that the K_ζ span $\mathscr{M}_s^\perp$. In both these extensions, formulas (A) and (B) hold for the first component of the image pair, but no formula for the second component was found.

The purpose of this note is to announce explicit formulas for the second components of the extended Fourier transforms. In particular, these formulas permit us to remove any assumption concerning the density of the functions $K_\zeta(z, t)$ in $\mathscr{M}_s^\perp$. Indications of the proofs are given; details will appear elsewhere.

2. **On transform** (A). We have

THEOREM 1. *Let σ be a continuous measure on* $[0, 2\pi]$. *Let s be defined by* (1) *and replace $\mathscr{U}$ by $\mathscr{U}_1$ in* (A). *The map $\mathscr{U}f = (\mathscr{U}_1 f, \mathscr{U}_2 f)$, where*

$$(\mathscr{U}_2 f)(t) = 2^{-1/2}\Delta(e^{it})\left[2\int_t^{2\pi} (f(\lambda) - f(t)e^{-\sigma([t,\lambda])})s_\lambda(e^{it})[s(e^{it})(1 - e^{i(t-\lambda)})]^{-1}\, d\sigma(\lambda) - s_t(0)f(t)/s(0)\right]$$

for all $f \in L^2(d\sigma)$ such that $f(\lambda)$ is continuous on $[t, t + \delta]$ (for some δ) and $e^{\sigma([0,\lambda])}f(\lambda)$ is differentiable at t, extends uniquely to a unitary operator (again denoted $\mathscr{U}$) of $L^2(d\sigma)$ onto $\mathscr{M}_s^\perp$. The extended $\mathscr{U}$ satisfies (2), *where $T: \mathscr{M}_s^\perp \to \mathscr{M}_s^\perp$ is defined by*

(7) $$T(f(z), g(t)) = P_{\mathscr{M}_s^\perp}(zf(z), e^{it}g(t))$$

and K, M are defined in (3).

As in [1, Lemma 1.1], the functions $c_\pi(\lambda) = e^{-\sigma([0,\lambda])}\chi_\eta(\lambda)$ span $L^2(d\sigma)$

(χ_η is the characteristic function of $[0, \eta]$). One computes $\mathscr{U}c_\eta = d_\eta$ and $(d_\eta, d_\mu) = (c_\eta, c_\mu)$, where

$$d_\eta(z, t) = 2^{-1/2}(1 - s_\eta(0)s_\eta(z), \ -s_\eta(0)\Delta_\eta(e^{it})s_\eta(e^{it})/s(e^{it}))$$

($\Delta_\eta = (1 - |s_\eta|^2)^{1/2}$). Also if $f \in L^2(d\sigma)$ satisfies the continuity and differentiability conditions of Theorem 1, one can approximate f by linear combinations of the c_η so that the integrands in the definition of $\mathscr{U}_2$ remain uniformly bounded on $[t, t + \delta]$. Thus, from $\mathscr{U}c_\eta = d_\eta$ and $(c_\eta, c_\mu) = (d_\eta, d_\mu)$, it follows that $\mathscr{U}$ has an isometric extension to all of $L^2(d\sigma)$. Next, one verifies that $\mathscr{U}M^*(I - K^*)c_\eta = T^*d_\eta$; (2) will follow if we can prove the extended $\mathscr{U}$ is onto. It is shown that the range of $\mathscr{U}$ is closed, contains the functions K_ζ ($|\zeta| < 1$), and is T^*-invariant. It follows that $\mathscr{U}$ is onto.

From Theorem 1 comes the following somewhat unexpected extension of Theorem 6.1 of [**1**].

COROLLARY 1. *The functions d_η span $\mathscr{M}_s^\perp$ (even when the functions K_ζ do not).*

3. **On transform** (B). Our second theorem is

THEOREM 2. *Let ν be any measure on* $[0, 2\pi]$. *Let s be defined by* (4), *and replace $\mathscr{U}$ by $\mathscr{U}_1$ in* (B). *The map $\mathscr{U}f = (\mathscr{U}_1 f, \mathscr{U}_2 f)$, where*

$$(\mathscr{U}_2 f)(x) = \Delta(e^{ix})(1 - s(e^{ix}))^{-1}\left[f(x) - \lim_{r \to 1-} (\mathscr{U}_1 f)(re^{ix})\right]$$

maps $L^2(d\nu)$ unitarily onto $\mathscr{M}_s^\perp$, and satisfies (5), *where T comes from* (7) *and H from* (6).

Let $P^2(d\nu)$ denote the L^2 closure of the polynomials in e^{ix}. That $\mathscr{U}$ maps P^2 unitarily onto the span of the K_ζ ($|\zeta| < 1$) in $\mathscr{M}_s^\perp$ comes essentially from [**2**], together with the computation of $\mathscr{U}_2$ of the appropriate functions in $P^2(d\nu)$.

If $f \perp P^2(d\nu)$, then $\mathscr{U}_1 f(z) = 0$ for every z, so $\mathscr{U}f$ is orthogonal to all K_ζ in $\mathscr{M}_s^\perp$. In addition, if $d\nu = w\,dx + d\nu_s$ is the Lebesgue decomposition of $d\nu$, it is not hard to see from (4) that $w = \Delta(e^{ix})^2|1 - s(e^{ix})|^{-2}$ and thus that $\mathscr{U}$ is isometric on $(P^2)^\perp$. $\mathscr{U}$ may be inverted directly to show it is onto. Another direct computation yields (5).

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