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FUNCTIONAL COMPOSITION ON SOBOLEV SPACES

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Introduction. This announcement concerns the study of conditions on functions $g(x, t_1, \ldots, t_m)$, $x \in \Omega \subset R_n$, under which the function g provides, via composition, a mapping from certain product Sobolev spaces $W_{r,q}(\Omega)^m$ to other Sobolev spaces $W_{1,p}(\Omega)$. Such questions are of interest in the study of nonlinear partial differential equations and elsewhere (see **[4]**, for example). Much of the analysis hinges on the use of a (seemingly new) notion of absolute continuity on tracks of absolutely continuous curves. This notion may also be useful elsewhere.

Statement of results. In what follows $\mathscr{H}_1$ denotes one-dimensional Hausdorff measure, $\mathscr{L}_n$ denotes n-dimensional Lebesgue measure, and R_k denotes k-dimensional Euclidean space.

Let $T \subset R_m$ be the track of an absolutely continuous curve. Then by results of Roger **[5]** and Federer **[1]**, for all points $y \in T$ with the exception of an $\mathscr{H}_1$-null set there is a unique pair of tangent directions to T at y, to be denoted by unit vectors θ_y, $-\theta_y$. Let $g: T \to R_1$ be defined on T. We say that g has a tangential derivative at y if, at y, T possesses a unique pair of tangent directions and if both the following relations hold for sequences $\{y_i\} \in T$:

$$\frac{\overline{yy_i}}{|y_i - y|} \to \theta_y \Rightarrow \frac{g(y_i) - g(y)}{|y_i - y|} \to \Lambda,$$

$$\frac{\overline{yy_i}}{|y_i - y|} \to -\theta_y \Rightarrow \frac{g(y_i) - g(y)}{|y_i - y|} \to -\Lambda, \qquad \Lambda \in R_1.$$

In this case the quantity $|\Lambda| \equiv D_T g(y)$ is called *the tangential derivative* of g at y. We shall say that g is *absolutely continuous* on T provided that $D_T g$ is defined $\mathscr{H}_1$-a.e. on T (in this case $D_T g$ is necessarily $\mathscr{H}_1$-measurable) and the following relation is satisfied

$$\text{(ac)} \qquad |g(y) - g(y')| \leqq \int_U D_T g(y)\, d\mathscr{H}_1 < \infty,$$

whenever $y, y' \in T$ and $U \subset T$ is a closed connected subset containing y and y'. Call a function $g: R_m \to R_1$ *fully absolutely continuous* provided

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that it is a Borel function and is absolutely continuous on all such tracks T.

The above notion of absolute continuity on a track coincides with the usual one when $m = 1$ and T is a real interval, and for arbitrary m shares many of the properties of the notion of absolute continuity for real functions on an interval. In particular, every locally Lipschitz function on R_m can be shown to be fully absolutely continuous. Moreover the following generalization of a classical chain rule of Vallée Poussin is available.

THEOREM 1. *Suppose that* $\boldsymbol{v} = (v_1, \ldots, v_m): I \to R_m$ *is absolutely continuous on the interval* I *and that* $g: R_m \to R_1$ *has a total differential at all points of the track* $T = \boldsymbol{v}(I)$ *except for an* $\mathscr{H}_1$*-null set, and is such that* $g|_T$ *is absolutely continuous on* T. *Then the composite function* $w = g(v_1, \ldots, v_m): I \to R_1$ *is absolutely continuous if and only if the following function is Lebesgue integrable on* I:

$$w_1 = \sum \frac{\partial g}{\partial t_j}(\boldsymbol{v})\dot{v}_j,$$

(all products being interpreted as zero wherever their second factor is zero). In that case

$$\dot{w} = w_1 \quad \textit{almost everywhere on } I.$$

Combining this result with a characterization, due to Gagliardo [2], of the Sobolev spaces $W_{1,q}(\Omega)$ (where $W_{1,q}(\Omega)$ is the class of L_q functions with L_q strong first derivatives on the domain $\Omega \subset R_n$), we obtain the following results.

THEOREM 2. *Let* Ω *be a domain in* R_n. *Suppose that* $g: R_m \to R_1$ *is fully absolutely continuous and that its points of nondifferentiability meet the track of every absolutely continuous curve in an* $\mathscr{H}_1$*-null set. Let* $\boldsymbol{u} = (u_1, \ldots, u_m)$ *where* $u_j \in W_{1,1}^{\mathrm{loc}}(\Omega)$, $1 \leqq j \leqq m$, *and set* $v = g(u_1, \ldots, u_m)$. *Then* $v \in W_{1,1}^{\mathrm{loc}}(\Omega)$ *if and only if the functions*

$$v_i = \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(\boldsymbol{u})\partial_i u_j, \qquad i = 1, \ldots, n,$$

belong to $L_1^{\mathrm{loc}}(\Omega)$, *where* ∂_i *denotes strong differentiation and where the products are to be interpreted as zero wherever their second factor is zero. In that case*

$$v_i = \partial_i v \qquad \mathscr{L}_n\textit{-a.e. in } \Omega, \quad i = 1, \ldots, n.$$

For the case $m = 1$, i.e. $g: R_1 \to R_1$ locally absolutely continuous in the usual sense, this result was obtained by Serrin [6]. Serrin's result stimulated our interest in obtaining the results stated above under assumptions which generalize the usual notion of absolute continuity.

REMARK. The need for some hypothesis on the nondifferentiability points of g is clear from the following example with $n = 1$. Let $g(t_1, \dots, t_m) = \max(t_1, \dots, t_m)$ and let $u_1(x) = \cdots = u_m(x) = x$, $\forall x \in (0, 1)$. Then the function v_1 is *nowhere* defined on $\Omega = (0, 1)$, while $\partial_1 v \equiv 1$.

On the other hand it is *possible* for functions g to provide mappings between Sobolev spaces even in the absence of chain rules. This aspect of the problem is studied in [**3**].

THEOREM 3. *Let Ω be a bounded domain in R_n possessing the cone property. Let g be a function satisfying the assumptions in Theorem 2 and denote $h_j = \partial g/\partial t_j$ (h_j is necessarily Borel measurable and defined $\mathscr{L}_n$-a.e.), $j = 1, \dots, m$. Given p, $1 \leqq p \leqq n$, suppose that for some q, $p < q < n$, the functions h_j determine, via composition, mappings which satisfy*:

$$h_j : L_{q^*}(\Omega)^m \to L_{q'}(\Omega) \qquad (j = 1, \dots, m), \quad \textit{with } q^* = \frac{nq}{n-q}, \; q' = \frac{pq}{q-p}.$$

Then g yields, via composition, a mapping which satisfies:

$$g : W_{1,q}(\Omega)^m \to W_{1,p}(\Omega).$$

Moreover, with $v = g(u_1, \dots, u_m)$ one has for $\boldsymbol{u} = (u_1, \dots, u_m) \in W_{1,q}(\Omega)^m$

$$\partial_i v = \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(\boldsymbol{u}) \partial_i u_j, \qquad i = 1, \dots, n,$$

the products being interpreted as zero wherever their second factor is zero.

REMARK. Such conditions on the h_j hold in particular when the h_j satisfy growth conditions of the following sort:

$$|h_j(\boldsymbol{t})| \leqq a + b|\boldsymbol{t}|^{\alpha}, \qquad \alpha = q^*/q', 1 \leqq j \leqq n.$$

We have the following type of result for higher order Sobolev spaces.

THEOREM 4. *Let Ω be a domain in R_n. Suppose that $g : R_{n+1} \to R_1$ is fully absolutely continuous and that its points of nondifferentiability meet the track of every absolutely continuous curve in a set whose projection onto the t_0-axis in R_{n+1} is a Lebesgue null set. Let $u \in W_{2,1}^{\mathrm{loc}}(\Omega)$ and set $v = g(u, \partial_1 u, \dots, \partial_n u)$. Then $v \in W_{1,1}^{\mathrm{loc}}(\Omega)$ if and only if the functions*

$$v_i = \frac{\partial g}{\partial t_0}(\boldsymbol{u}) \partial_i u + \sum_{j=1}^{n} \frac{\partial g}{\partial t_j}(\boldsymbol{u}) \partial_i \partial_j u, \qquad i = 1, \dots, n,$$

belong to $L_1^{\mathrm{loc}}(\Omega)$, where the products are to be interpreted as zero wherever their second factor is zero. In that case we have

$$v_i = \partial_i v \qquad \mathscr{L}_n\text{- a.e. in } \Omega, \quad i = 1, \dots, n.$$

This can be exploited to obtain conditions on the functions $h_j = \partial g/\partial t_j$,

$0 \leqq j \leqq n$, as mappings between spaces $L_\rho(\Omega)$, which ensure that g yields, via composition, a mapping of Sobolev spaces of the following type:

$$g: W_{2,q}(\Omega) \to W_{1,p}(\Omega), \qquad 1 \leqq p < q \leqq n/2.$$

The results, which are analogues of Theorem 3 are omitted here.

We sketch the proof of Theorem 2. By Gagliardo's result [**2**] we are able to restrict attention to functions u_j which are locally absolutely continuous (as a function of one variable) on $\tau \cap \Omega$ for almost all lines τ parallel to any axis x_i in R_n. We may then utilize Theorem 1 to show that v is absolutely continuous on subintervals of $\tau \cap \Omega$, when and only when $\partial v/\partial x_i$ coincides almost everywhere with v_i on such subintervals. We then utilize Gagliardo's characterization, once more, to deduce that $v \in W_{1,1}^{\mathrm{loc}}(\Omega)$ if and only if $v_i \in L_1^{\mathrm{loc}}(\Omega)$, $1 \leqq i \leqq n$.

For functions g defined on $\Omega \times R_m$ we have closely analogous results to the above provided that g is locally Lipschitz (rather than merely fully absolutely continuous) on $\Omega \times R_m$. As a sample result of this type we mention the following.

THEOREM 5. *Let Ω be a bounded domain in R_n possessing the cone property. Suppose that $g: \Omega \times R_m \to R_1$ is locally Lipschitz on $\Omega \times R_m$. Suppose also that the set of nondifferentiability points of g has the property that its projection on R_m meets the track of every absolutely continuous curve in R_m in an $\mathscr{H}_1$-null set and, in addition, that for all $(\boldsymbol{x}, \boldsymbol{t}) \in \Omega \times R_m$ where the derivatives mentioned below exist, they satisfy the growth conditions:*

$$\left|\frac{\partial g}{\partial x_i}(\boldsymbol{x}, \boldsymbol{t})\right| \leqq a_1(\boldsymbol{x}) + b_1|\boldsymbol{t}|^\nu, \qquad i = 1, \ldots, n,$$

$$\left|\frac{\partial g}{\partial t_j}(\boldsymbol{x}, \boldsymbol{t})\right| \leqq a_2(\boldsymbol{x}) + b_2|\boldsymbol{t}|^{\nu-1}, \qquad j = 1, \ldots, m,$$

where $\nu \geqq 1$ is a fixed number; $a_1 \in L_p(\Omega)$ for some $1 < p < n$; $a_2 \in L_r(\Omega)$ with $r = (p\nu/(\nu - 1)) \cdot (n/(n - p))$ [$n = \infty$ for $\nu = 1$]; and $|\boldsymbol{t}| = |t_1| + \cdots + |t_m|$. Then g yields, via composition, a mapping of a Sobolev space $W_{1,q}(\Omega)^m$ into $W_{1,p}(\Omega)$:

$$g: W_{1,q}(\Omega)^m \to W_{1,p}(\Omega) \quad \text{with} \quad q = \nu p \cdot n/(n + (\nu - 1)p),$$

and for all $\boldsymbol{u} = (u_1, \ldots, u_m) \in W_{1,q}(\Omega)^m$,

$$\|g(x, u_1(x), \ldots, u_m(x))\|_{W_{1,p}(\Omega)} \leqq \text{const}[1 + \|(u_1, \ldots, u_m)\|^\nu_{W_{1,q}(\Omega)^m}],$$

where the constant depends on $\Omega, a_1, a_2, b_1, b_2$ and $g_0(x) \equiv g(x, 0, \ldots, 0)$, but not on $\boldsymbol{u}$. Moreover this mapping is demicontinuous, i.e. continuous from the strong to the weak topology.

Proofs and related results will appear elsewhere.

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