BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

ON EXTENSIONS OF FUNDAMENTAL GROUPS OF SURFACES AND RELATED GROUPS

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Communicated by Fred Gehring, May 24, 1971

1. The aim of this paper is to sketch a proof of a conjecture of A. Karrass and D. Solitar raised at the American Mathematical Society meeting in Urbana, October 1970: *A finite torsionfree extension of the fundamental group of a closed surface is isomorphic to the fundamental group of a closed surface.* By using the same methods we can prove a similar result for fundamental groups of Seifert fibre spaces.

It is natural that the proofs are quite different for the (euclidean) cases of the torus and Klein bottle and the other (noneuclidean) surfaces. The proof in the euclidean case could be extracted from results on crystallographic groups of Bieberbach, Burckhardt and Frobenius, but I shall sketch an elementary proof here. In the proof for the noneuclidean cases we shall use the theorem of J. Nielsen and S. Kravetz that any finite subgroup of the outer automorphism group of the fundamental group of a closed surface can be realized by a finite group of homeomorphisms [1].[1]

2. THEOREM 1. *Let $\mathfrak{G}$ be a group without torsion, $\mathfrak{F}$ a subgroup of finite index isomorphic to the fundamental group of a closed "noneuclidean" orientable surface (i.e. the genus is >1 in the orientable case, >2 in the nonorientable). Then $\mathfrak{G}$ is isomorphic to the fundamental group of a closed noneuclidean surface.*

From Kneser's formula [6, p. 50] it follows

COROLLARY. *If $\mathfrak{G}$ and $\mathfrak{F}$ are the fundamental groups of orientable surfaces of genus g and f, resp., and if $\mathfrak{F}$ is a subgroup of $\mathfrak{G}$ with $c=[\mathfrak{G}:\mathfrak{F}]$, then $g-1=c(f-1)$.*

In the proof of Theorem 1 we may restrict ourselves to the case

AMS 1970 *subject classifications*. Primary 20F25, 20H10, 20H15, 55F55.
Key words and phrases. Fundamental groups of surfaces, fuchsian groups, crystallographic groups, group extensions, and Seifert fibre spaces.

[1] ADDED IN PROOF. Recently it was found out that the theorem about negative curvature of the Teichmüller space in [1] is not true. Therefore [1, Theorem 5.2] is not known to be valid and the Theorems 1, 1′ and 5 of this article are still open. In the special cases of finite extensions by solvable groups the theorems will follow from the fixed point theorem of P. A. Smith (Ann. of Math. **35** (1934), 572–578).

where $\mathfrak{F}$ is normal in $\mathfrak{G}$ and isomorphic to the fundamental group of an orientable surface F. From the theorems of J. Nielsen [6, p. 111], R. Baer [6, p. 133] and J. Nielsen-S. Kravetz ([2], [1, Theorem 5.2]) it follows that the quotient group $\mathfrak{G}/\mathfrak{F}$ may be represented by a group of homeomorphisms on F. By returning to the universal covering of F, we find $\mathfrak{G}$ to be isomorphic to a fuchsian group with compact fundamental region. As $\mathfrak{G}$ is torsionfree, $\mathfrak{G}$ is isomorphic to a surface group. By the same arguments the result can be generalized to

THEOREM 1′. *Let $\mathfrak{F}$ be a fuchsian group with compact fundamental region and $\mathfrak{G}$ a finite extension of $\mathfrak{F}$ with the following property:*

(*) *if $g \in \mathfrak{G}$ and $g^{-1}xg = x$ holds for all $x \in \mathfrak{F}$, then $g = 1$.*

Then $\mathfrak{G}$ is a fuchsian group with compact fundamental region.

REMARK. The property (*) does not hold in the euclidean case; it is used in the application of the Baer theorem.

3. THEOREM 2. *Let $\mathfrak{G}$ be a torsionfree finite extension of $\mathfrak{T} = \mathbf{Z}^n$ (n a positive integer) such that $g^{-1}xg = x$ for all $x \in \mathfrak{T}$ and all $g \in \mathfrak{G}$. Then $\mathfrak{G}$ is isomorphic to $\mathbf{Z}^n$.*

The theorem is known in the theory of crystallographic groups, [4, Theorem 3.2.9]; another simple proof can be based on the

LEMMA. *For $x, y \in \mathfrak{G}$ let l, m, n denote the orders of the cosets $x\mathfrak{T}$, $y\mathfrak{T}$ and $xy\mathfrak{T}$. Then n divides the least common multiple k of l and m, and $(xy)^k = x^k y^k$.*

THEOREM 3. *A torsionfree finite extension $\mathfrak{G}$ of $\mathbf{Z}$ is isomorphic to $\mathbf{Z}$; a torsionfree finite extension of $\mathbf{Z}^2$ is isomorphic to $\mathbf{Z}^2$ or to the Klein bottle group $\{a, b; a^2b^2 = 1\}$.*

This is an easy consequence of Theorem 2, because the automorphisms of finite order of $\mathbf{Z}$ or those of $\mathbf{Z}^2$ with determinant $+1$ will not fix an element different from 0. In the case $\mathbf{Z}^2$, let $\mathfrak{T}$ denote a maximal abelian subgroup of $\mathfrak{G}$. By Theorem 2, $\mathfrak{T}$ is isomorphic to $\mathbf{Z}^2$. $\mathfrak{G}/\mathfrak{T}$ is either trivial or $\mathbf{Z}_2$. In the latter case $\mathfrak{G}$ is isomorphic to the Klein bottle group.

4. THEOREM 4. *Let $\mathfrak{G}$ be a torsionfree extension of $\mathfrak{H} = \mathbf{Z}$ by a fuchsian group with compact fundamental region acting trivially on $\mathfrak{H}$. Then $\mathfrak{G}$ is isomorphic to the fundamental group of a closed Seifert fibre space.*

(For a definition and literature see [3]. Let us call a Seifert fibre space *sufficiently large*, if a fuchsian group is obtained by factoring out the maximal cyclic normal subgroup.)

$\mathfrak{G}/\mathfrak{H}$ will have the form

$$\overline{\mathfrak{G}} = \left\{ \bar{s}_1, \cdots, \bar{s}_m, \bar{t}_1, \bar{u}_1, \cdots, \bar{t}_g, \bar{u}_g; \right.$$
$$\left. \bar{s}_1^{a_1} = \cdots = \bar{s}_m^{a_m} = \bar{s}_1 \cdots \bar{s}_m \prod_{i=1}^{g} [\bar{t}_1, \bar{u}_1] = 1 \right\}.$$

We fix elements $s_1, \cdots, u_g$ in $\mathfrak{G}$ that are mapped to $\bar{s}_1, \cdots, \bar{u}_g$, resp. Let h denote a generator for $\mathfrak{H}$. Then there are relations $s_i^{a_i} = h^{\alpha_i}$, $(a_i, \alpha_i) = 1$ and $s_1 \cdots s_m \prod_{i=1}^{g} [t_i, u_i] = h^{\alpha}$. The group

$$\tilde{\mathfrak{G}} = \left\{ \tilde{h}, \tilde{s}_1, \cdots, \tilde{s}_m, \tilde{t}_1, \tilde{u}_1, \cdots, \tilde{t}_g, \tilde{u}_g; \right.$$
$$\left. \tilde{s}_i^{a_i} = \tilde{h}^{\alpha_i}, \tilde{s}_i \cdots \tilde{s}_m \prod_{i=1}^{g} [\tilde{t}_i, \tilde{u}_i] = \tilde{h}^{\alpha} \right\}$$

is the fundamental group of a Seifert fibre space [3]. We may map $\tilde{\mathfrak{G}}$ onto $\overline{\mathfrak{G}}$ letter by letter. By $\tilde{h} \mapsto h, \cdots, \tilde{u}_g \mapsto u_g$ is defined an homomorphism $\tilde{\mathfrak{G}} \to \mathfrak{G}$ inducing the identity on $\overline{\mathfrak{G}}$. As h has infinite order and the kernel of $\mathfrak{G} \to \overline{\mathfrak{G}}$ is generated by h, the homomorphism $\tilde{\mathfrak{G}} \to \mathfrak{G}$ turns out to be an isomorphism.

REMARK. From [5] it follows that Theorem 4 can be generalized to $\boldsymbol{Z}^n$ instead of $\boldsymbol{Z}$, and to the case of nontrivial action.

5. THEOREM 5. *A torsionfree finite extension $\mathfrak{G}$ of the fundamental group $\mathfrak{F}$ of a sufficiently large Seifert fibre space is isomorphic to the fundamental group of a Seifert fibre space.*

Let $\mathfrak{H}$ denote the cyclic normal subgroup of $\mathfrak{F}$. As $\mathfrak{H}$ is characteristic in $\mathfrak{F}$, $\mathfrak{H}$ is normal in $\mathfrak{G}$. The group $\mathfrak{S} = \{x \in \mathfrak{G} : f^{-1}x^{-1}fx \in \mathfrak{H}$ for all $f \in \mathfrak{F}\}$ is normal in $\mathfrak{G}$ and contains $\mathfrak{H}$. $[\mathfrak{S} : \mathfrak{H}]$ divides $[\mathfrak{G} : \mathfrak{F}]$. By Theorem 3, $\mathfrak{S}$ turns out to be cyclic. $\tilde{\mathfrak{G}} = \mathfrak{G}/\mathfrak{S}$ can be considered as an extension of $\mathfrak{F}/\mathfrak{H}$ such that (*) holds. Therefore $\tilde{\mathfrak{G}}$ is fuchsian by Theorem 1'. From Theorem 4 it follows Theorem 5.

COROLLARY.[2] *If a sufficiently large Seifert fibre space is a finite covering of another space, this has to be a sufficiently large Seifert fibre space.*

6. For generalizations to higher dimensions one has to consider crystallographic groups. The family of torsionfree crystallographic

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[2] This result also was obtained by W. Jaco.

groups itself is closed under finite torsionfree extensions. The family of groups isomorphic to fundamental groups of Seifert fibre spaces in higher dimensions [5] has to be enlarged to get an analogue to Theorem 5.

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