BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 6, November 1971

EXISTENCE OF POLYNOMIAL IDENTITIES IN $A \otimes_F B$

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Communicated by M. H. Protter, April 28, 1971

ABSTRACT. The following theorem is proved: If A, B are PI-algebras over a field F, then $A \otimes_F B$ is also a PI-algebra.

Let F be a field, A and B two PI-algebras (i.e., algebras satisfying a polynomial identity) over F. The problem whether also $A \otimes_F B$ satisfies a polynomial identity has been open for some time [**1**, p. 228]. We have proved that if A and B are PI-algebras, then $A \otimes_F B$ is indeed a PI-algebra. A very brief outline of the proof is given here, and the details of the proof will appear elsewhere.

Let $\{x\}$ be an infinite set of noncommutative indeterminates over F, and let $F[x]$ be the free ring in $\{x\}$ over F. Let $\{x_1, x_2, \cdots\} = \{x_\nu\} \subseteq \{x\}$ be a fixed countable sequence of indeterminates from $\{x\}$. Let S_n denote the group of all permutations of $\{1, \cdots, n\}$ and let

$$V_n = \operatorname{span}\{x_{\sigma_1} \cdots x_{\sigma_n} \mid \sigma \in S_n\}$$

be the $n!$ dimensional vector space, spanned by the $n!$ monomials $x_{\sigma_1} \cdots x_{\sigma_n}$ $(\sigma \in S_n)$ in $x_1, \cdots, x_n$.

An ideal $Q \subseteq F[x]$ is a T-ideal if $f(x_1, \cdots, x_n) \in Q$ and $g_1, \cdots, g_n \in F[x]$ implies that $f(g_1, \cdots, g_n) \in Q$. It is well known [**1**, p. 234] that the set of all identities of a PI-algebra is a T-ideal. Let Q be the T-ideal of identities of a PI-algebra A. For each integer $0<n$, define $d_n = \dim(V_n/(Q \wedge V_n))$. We call $\{d_\nu\}$ "the sequence of codimensions" of Q (or A). Codimensions play an important role in the proof that $A \otimes_F B$ is a PI-algebra.

It follows from the definition of d_n that there exist d_n monomials $M_1(x_1, \cdots, x_n), \cdots, M_{d_n}(x_1, \cdots, x_n)$ which span V_n modulo Q, i.e., for each $\sigma \in S_n$ there exist coefficients $\phi_i(\sigma) \in F$, $1 \leqq i \leqq d_n$, such that

$$M_\sigma(x) = x_{\sigma_1} \cdots x_{\sigma_n} \equiv \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(x) \pmod{Q}.$$

AMS 1970 *subject classifications*. Primary 16A38.

[1] This paper was written while the author was doing his Ph.D. thesis at the Hebrew University of Jerusalem under the kind supervision of Professor S. A. Amitsur, to whom he wishes to express his warm thanks.

Since Q is the ideal of identities of A, it follows that for any substitution $a_1, \cdots, a_n \in A$ we have

$$a_{\sigma_1} \cdots a_{\sigma_n} = \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a_1, \cdots, a_n).$$

We now prove

THEOREM 1. *Let A and B be two PI-algebras with $\{d_\nu\}$, $\{h_\nu\}$ the corresponding sequences of codimensions. If there exists an integer $0 < n$ such that $d_n h_n < n!$, then $A \otimes_F B$ satisfies a nontrivial identity of degree n.*

PROOF. Let $M_1(x), \cdots, M_{d_n}(x)$, $\phi_i(\sigma) \in F$, $1 \leqq i \leqq d_n$, $\sigma \in S_n$, be monomials and coefficients such that for all $a_1, \cdots, a_n \in A$ and $\sigma \in S_n$, $a_{\sigma_1} \cdots a_{\sigma_n} = \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a)$. Let, similarly, $N_j(x)$, $\psi_j(\sigma)$, $1 \leqq j < h_n$, $\sigma \in S_n$, be monomials and coefficients such that for all $\sigma \in S_n$ and $b_1, \cdots, b_n \in B$,

$$b_{\sigma_1} \cdots b_{\sigma_n} = \sum_{j=1}^{h_n} \psi_j(\sigma) N_j(b).$$

Write

$$(*) \qquad f(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma_1} \cdots x_{\sigma_n}$$

with the α_σ undetermined coefficients. Now

$$\begin{aligned} f(a_1 \otimes b_1, \cdots, a_n \otimes b_n) &= \sum_{\sigma \in S_n} \alpha_\sigma (a_{\sigma_1} \cdots a_{\sigma_n}) \otimes (b_{\sigma_1} \cdots b_{\sigma_n}) \\ &= \sum_{i=1}^{d_n} \sum_{j=1}^{h_n} \left(\sum_{\sigma \in S_n} \phi_i(\sigma) \psi_j(\sigma) \alpha_\sigma \right) M_i(a) \otimes N_j(b). \end{aligned}$$

Since $d_n h_n < n!$, there exists a nontrivial solution $\{\alpha_\sigma\}_{\sigma \in S_n}$ for the $h_n d_n$ homogeneous linear equations $\sum_{\sigma \in S_n} \phi_i(\sigma) \psi_j(\sigma) \alpha_\sigma = 0$ in $n!$ indeterminates. Clearly the α_σ yield (for (*)) a nontrivial identity $f(x_1, \cdots, x_n)$ for $A \otimes_F B$.

The second and the difficult step in the proof that $A \otimes B$ is a PI-algebra is:

THEOREM 2. *Let $\{d_n\}$ be the sequence of codimensions of an arbitrary PI-algebra A. Then there exists a positive real number k such that for all $n \in N$, $d_n \leqq k^n$. (We actually prove if A satisfies an identity of degree d, then $k \leqq 3 \cdot 4^{d-3}$.)*

The proof of Theorem 2 is complicated and will be given elsewhere.

It is a combinatorial proof, and has nothing to do with the structure of the algebra.

Now, let A, B be two PI-algebras with $\{d_\nu\}$, $\{h_\nu\}$ their corresponding sequences of codimensions. Let k, l be such that for all n, $d_n \leqq k^n$, $h_n \leqq l^n$. Let n be such that $(k\cdot l)^n < n!$. Then, by Theorem 1, $A\otimes_F B$ satisfies an identity of degree n.

References

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