

CONVENIENT CATEGORIES OF TOPOLOGICAL ALGEBRAS

BY EDUARDO J. DUBUC AND HORACIO PORTA¹

Communicated by Joseph Rotman, April 22, 1971

Introduction. Concrete associative algebras with a topology have long arisen in mathematical practice; thus, a notion of topological space with algebraic operations making it an associative algebra was in order. The subject naturally evolved into the present general theory of abstract topological algebras [5]. Classes of such objects (together with continuous maps respecting the algebraic structure) form categories which, understandably, do not share some important properties of their purely algebraic analogues. Specifically, *their relation with the base category \mathbf{S} of sets is not satisfactory*. This is essentially due to the fact that taking forgetful functors into \mathbf{S} is forgetting too much. Also, the set of morphisms between any two such algebras naturally carries a topology which is inherited from the topologies of the algebras, and which is not taken into account (it is ignored) by the representable functors landing in \mathbf{S} .

The category of topological spaces is actually the natural *base* category (that is, the place where the forgetful and representable functors land) for a categorical approach to the study of classes of topological algebras. However, this category is not “set-like” enough to make such an approach possible.

Categories which, like \mathbf{S} , have enough structure to serve as base categories have been recognized by category theorists during the sixties ([1], [4]) when the concept of *closed* category was developed. Compactly generated topological spaces form such a convenient (closed) category [8].

We introduce here a systematic treatment of categories of topological algebras considered as categories based on the category \mathbf{K} of compactly generated Hausdorff spaces.

This leads to the definition of *K -topological algebras*. Roughly, a K -topological algebra is a complex algebra with a topology making the operations continuous when restricted to compact subsets. This is a broad class of algebras, containing *all* algebras with jointly continuous product, but failing to contain some topological algebras with

AMS 1969 subject classifications. Primary 1810, 4650; Secondary 4625, 4660.

Key words and phrases. Topological algebras, enriched category theory, Kelley spaces.

¹ Research of the second author supported by NSF GP 20431

discontinuous, separately continuous, product. However, some interesting examples of the latter actually are K -topological algebras, as in the case of von Neumann algebras with the strong topology.

Our approach allows us to treat topologized algebraic structures in a *purely algebraic* way, the topological information being carried automatically due to the closed structure of \mathbf{K} .

We obtain functional representations of certain locally m -convex algebras (considered as K -topological algebras), generalizing the Gelfand representation of commutative C^* -algebras. This is done by interpreting functional representations within the general framework of a duality machinery.

Some examples of interest to analysts are listed; proofs will appear in [3].

K -topological algebras. For a Hausdorff topological space X define KeX to be the space $KeX = \text{colim}_{K \subset X} K$, where K runs over all the compact sets of X (colim = direct limit). KeX is the K -ification of X , and X is called a *Kelley space* if $X = KeX$. The identity map $KeX \rightarrow X$ is clearly continuous. \mathbf{K} will denote the category of all Kelley spaces and continuous maps; \mathbf{K} is a (full) subcategory of the category $\mathbf{Top2}$ of all Hausdorff spaces and continuous maps, and $Ke: \mathbf{Top2} \rightarrow \mathbf{K}$ is a reflection (=right adjoint to the inclusion $\mathbf{K} \rightarrow \mathbf{Top2}$). For $X, Y \in \mathbf{K}$, the space of continuous maps $X \rightarrow Y$ with the compact-open topology is a Hausdorff space; denote by $\mathbf{K}(X, Y)$ its kellyfication. Then \mathbf{K} is a symmetric monoidal, closed, complete and cocomplete category, with tensor product given by the (categorical) product $X \boxtimes Y = Ke(X \times Y)$ ($X \times Y$ = cartesian product in $\mathbf{Top2}$). Clearly the field C of complex numbers with its ordinary topology is an object of \mathbf{K} .

DEFINITION 1. *A K -topological algebra is an algebra over C in \mathbf{K} . That is, it is a Kelley space X together with maps in \mathbf{K} : $+: X \boxtimes X \rightarrow X$, $0: 1 \rightarrow X$, $-: X \rightarrow X$, $\cdot: X \boxtimes X \rightarrow X$, and $\cdot: C \times X \rightarrow X$, satisfying the standard relations expressing associativity, commutativity (for $+$), etc. Thus, a K -topological algebra is defined within \mathbf{K} as the ordinary concept of "associative algebra" defined within \mathbf{S} .*

Examples. We give a list of examples of algebras whose kellyfications are K -topological algebras:

- (a) all topological algebras with jointly continuous product;
- (b) the following algebras with separately continuous product:
 - (i) von Neumann algebras with the strong topology;
 - (ii) $L^\infty(\mu)$, μ a measure, with the w^* -topology as dual of $L^1(\mu)$;
 - (iii) the algebra of bounded holomorphic functions on the unit disc with the strict topology;

(iv) the convolution algebra of all bounded measures on a compact group, with the vague topology,

(c) the algebra A consisting of all continuous complex functions on the interval of ordinals $[1, \Omega]$ ($\Omega = 1$ st uncountable ordinal) with the product $f \cdot g = f(\Omega)g(\Omega)e$, where e is the function $e(a) = 1$ for each ordinal $1 \leq a \leq \Omega$ and the topology of uniform convergence on the initial segments $[1, a]$, $a < \Omega$: observe that this product is *not* separately continuous.

Categories of K -topological algebras. A homomorphism between K -topological algebras A, B is a map $|A| \rightarrow |B|$ in \mathbf{K} commuting with the algebraic operations, where we denote by $|A|$, resp. $|B|$, the Kelley space supporting the algebra A , resp. B . The kellyfication of the compact-open topology on the set of homomorphisms between A and B defines an object $A(A, B) \in \mathbf{K}$, and the class of K -topological algebras and homomorphisms is a \mathbf{K} -category to be denoted here by \mathbf{A} . \mathbf{A}' will denote the subcategory of algebras having an identity and homomorphisms preserving it. The forgetful functor $\mathbf{A} \rightarrow \mathbf{K}$ defined by $A \rightarrow |A|$ is a \mathbf{K} -functor. Moreover we have (with the terminology of [2]):

THEOREM 2 (a). *The \mathbf{K} -categories \mathbf{A} and \mathbf{A}' are \mathbf{K} -complete, \mathbf{K} -cocomplete (in particular tensored and cotensored).*

(b) *The forgetful functors $\mathbf{A} \rightarrow \mathbf{K}$ and $\mathbf{A}' \rightarrow \mathbf{K}$ ($A \rightarrow |A|$) have \mathbf{K} -left adjoints, and in fact, are \mathbf{K} -monadic.*

(c) *The inclusion $\mathbf{A}' \rightarrow \mathbf{A}$ has a \mathbf{K} -left adjoint.*

It is easy to see that the limits in \mathbf{A} are the algebraic limits (pointwise operations) with the K -ifications of the limit topologies. The cotensor of a Kelley space X with a K -topological algebra A is the algebra of continuous functions $X \rightarrow A$ with pointwise operations and the K -ification of the compact-open topology. In particular, when $A = C$, the cotensor gives a categorical characterization for the algebra $C(X)$ of all complex continuous functions on X (again, with the K -ification of the topology of uniform convergence on compact sets of X). There are no simple descriptions of the dual concepts (colimits and tensors). Observe that Theorem 2(b) means that there is a free K -topological algebra over an arbitrary Kelley space, and Theorem 2(c) means that it is always possible to embed a K -topological algebra into a K -topological algebra with identity.

Duality and functional representations. Denote by $\bar{A}(X, A) \in \mathbf{A}'$ the cotensor of $X \in \mathbf{K}$ with $A \in \mathbf{A}'$ (cf. [2]), and let $T = (T', \eta', \mu')$ be the \mathbf{K} -monad in \mathbf{A}' defined by the pair of adjoints $\bar{A}'(-, C), A(-, C)$

so that $TA = \overline{A}(A(A, C), C)$. The unit $\eta'_A: A \rightarrow TA$ is the Fourier-Gelfand transformation (whose continuity for any K -topological algebra follows automatically due to the closed structure of K). The principal result in the classical functional representation theory of Gelfand means that $A \approx T'A$ via η'_A for any commutative C^* -algebra with unit A .

THEOREM 3. *There is a K -complete K -category B and a K -faithful K -functor $L: B \rightarrow A'$ which (strictly) preserves cotensors and K -limits. Furthermore*

- (a) *there is a unique object $C \in B$ such that $C = LC$;*
- (b) *C is a K -codense cogenerator of B , that is, for all $B \in B$, we have $B \approx \overline{B}(B(B, C), C)$;*
- (c) *given any $A \in A'$ such that $T'A \approx A$ via η'_A , there is a unique object $B \in B$ such that $LB = A$, and moreover, for any other $B \in B$, $B(B, B') \approx A'(A, LB')$ via L .*

THEOREM 4. *For an object $A \in A'$ the following are equivalent:*

- (a) *$A = \overline{A}'(X, C)$ for some Kelley space X ;*
- (b) *A is a limit in A' of commutative C^* -algebras with identity;*
- (c) *there is a complete commutative locally m -convex algebra M (cf. [7]) with involution $x \rightarrow x^*$ and a defining family of submultiplicative seminorms $\{p\}$ satisfying $p(x \cdot x^*) = p(x)^2$ for each $x \in M$, and such that $A = KeM$.*

The equality $KeM = \overline{A}'(X, C)$ (M as in Theorem 4(c)) obtained above is, of course, an algebraic equality as well as a topological equality. However, if the topologies are not K -ified, the continuity in one direction may be lost. Concretely, we have: for each algebra M as in Theorem 4(c) there is a Kelley space X and a map $S: C(X) \rightarrow M$ from the algebra $C(X)$ of complex continuous functions on X , into M , such that S is an algebraic isomorphism, S is continuous for the compact-open topology on $C(X)$ and $S^{-1}: M \rightarrow C(X)$ is continuous on compact sets.

Theorem 4 is actually a corollary of Theorem 3, which is a purely categorical result: the space X associated to an algebra M is constructed, and the equality $KeM = \overline{A}'(X, C)$ is proven by the use of a purely categorical machinery. We point out that this space X does not coincide in general, even as a set, with other spaces devised for the same purpose [6].

REFERENCES

1. J. Bénabou, *Catégories relatives*, C. R. Acad. Sci. Paris **260** (1965), 3824–3827. MR 31 #1284.

2. E. J. Dubuc, *Kan extensions in enriched category theory*, Lecture Notes in Math., no. 145, Springer-Verlag, Berlin and New York, 1970.
3. E. J. Dubuc and Horacio Porta, *Convenient categories of topological algebras, and their duality theory*, J. Pure Appl. Algebra (to appear).
4. S. Eilenberg and G. M. Kelly, *Closed categories*, Proc. Conference Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 421–562. MR 37 #1432.
5. A. Guichardet, *Special topics in topological algebras*, Gordon and Breach, New York, 1968. MR 39 #4673.
6. C. Sh. Hsia, *Seminormed Banach rings with involution*, Izv. Akad. Nauk SSSR Ser. Mat. 23 (1959), 509–528. (Russian)
7. E. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. No. 11 (1952), 79 pp. MR 14, 482.
8. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. 14 (1967), 133–152. MR 35 #970.

DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA

UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

Current address: University of Rochester, Rochester, New York 14627