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FINITE SIMPLE GROUPS OF LOW 2-RANK AND THE FAMILIES $G_2(q)$, $D_4^2(q)$, q ODD[1]

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ABSTRACT. In §1, we describe the presently known finite simple groups of 2-ranks 3 and 4 and discuss in outline the existing methods of characterizing such groups by means of their Sylow 2-subgroups. In the balance of the paper, we illustrate these techniques concretely by classifying all simple groups whose Sylow 2-subgroups are isomorphic to those of $G_2(q)$ for some odd q.

1. Simple groups of low 2-rank. The *2-rank* of a group G is, by definition, the maximum rank of an abelian 2-subgroup of G. Thus a group of 2-rank 0 is of odd order and so is solvable by the Feit-Thompson theorem [59]. A group of 2-rank 1 has either cyclic or generalized quaternion Sylow 2-subgroups and so is not simple (assuming it is of composite order) by Burnside's transfer theorem and a theorem of Brauer and Suzuki [58]. Recently it has been shown by Alperin, Brauer, and Gorenstein [56], using various previously established classification theorems, that the only simple groups of 2-rank 2 are the groups

$$L_2(q),\ L_3(q),\ U_3(q),\quad q \text{ odd},\qquad U_3(4),\ A_7, \text{ and } M_{11}.$$

Work on the determination of simple groups of 2-rank 3 and 4 has only recently begun. As in the case of groups of 2-rank 2, the analysis divides into two major parts:

A. Determination of the possible Sylow 2-subgroups of a simple group of 2-rank 3 or 4.

B. Classification of simple groups having Sylow 2-subgroups of each of the types specified under A.

At the present time, work on problem A is only just beginning. However, whatever the outcome of such an analysis, the list of possible Sylow 2-subgroups will obviously include those of all the presently known simple groups of 2-rank 3 or 4. Thus it will be instructive to list these groups.

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In the table below the groups $Re(q)$, q an odd power of 3, J_1, J_2, J_3, M^c, H-S, L, and C_3 denote respectively the Ree groups $G_2^1(q)$ of characteristic 3, Janko's three groups of respective orders $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, $2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$, McLaughlin's group of order $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$, the Higman-Sims group of order $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, Lyons group of order $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$, the existence of which has recently been established by Sims [8], and the smallest Conway group C_3 of order $2^{10} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. Moreover, $D_4^2(q)$ designates the "triality" twisted $D_4(q)$.

In addition, we have placed a $+$ or $-$ in the final column according as the given group or family of groups has or has not been characterized at the present time in terms of the isomorphism class of its Sylow 2-subgroups.[2]

A few comments are in order. First, the groups $Re(q)$ have not quite been characterized. Walter [39] has shown only that a simple group G with abelian Sylow 2-subgroups of rank 3 is either isomorphic to $L_2(8)$, J_1, or is of "Ree type." Although groups of Ree type have been studied intensively, it is still unsettled whether the groups $Re(q)$ are the only such groups. However, the subgroup structure of the groups of Ree type so closely resembles that of the Ree groups themselves that one acts, in practice, as though the Ree groups have been classified, even though this is not, strictly speaking, correct.

Furthermore, considerable work has been done on simple groups with Sylow 2-subgroups of type $D_{2^n} \int Z_2$, $n \geqq 3$, which occur as Sylow 2-subgroups of $L_4(q)$, $q \equiv 3 \pmod 8$ and $U_4(q)$, $q \equiv 5 \pmod 8$ by D. Mason of Cambridge University and it is likely that also these groups will soon be characterized by their Sylow 2-subgroups.

Finally, it can be anticipated that the list of 2-groups S to be determined under A will also include those of the form $S = S_1 \times S_2$, where S_1, S_2 are groups which occur as Sylow 2-subgroups of simple groups of 2-rank 2. Thus S_1, S_2 are either dihedral, quasi-dihedral, wreathed, or of type $U_3(4)$. The reason for this is that the nature of the analysis that will be carried out under A will most likely not utilize the full force of the assumption of the simplicity of G and so at this stage one will not be able to rule out the possibility that G might have the structure of a direct product of simple groups.

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[2] ADDED IN PROOF. Work on similar characterizations of many of the groups in the table is in progress and in some cases has been completed. Mason is now treating $L_4(q)$, $U_4(q)$ for all odd q, while R. Solomon of Yale University is working on C_3. $U_3(8)$ and $U_3(16)$ have been considered by Collins and Griess independently and $H = S$ by Gorenstein and Harris jointly. Thus only the families $L_5(q)$, $U_5(q)$, $PS_p(6, q)$, q odd, $L_6(q)$, $q \equiv -1 \pmod 4$ and $U_6(q)$, $q \equiv 1 \pmod 4$ remain to be investigated.

RANK 3

Odd Characteristic	$G_2(q)$, $D_4^2(q)$, $Re\ (q)$	+
Even Characteristic	$L_2(8)$, $Sz(8)$, $U_3(8)$	+ −
Sporadic Groups	J_1, M_{12}	+

RANK 4

Odd Characteristic	$PSp(4, q)$ $L_4(q)$, $U_4(q)$, $L_5(q)$, $U_5(q)$, $PSp(6, q)$ $L_6(q)$, $q\equiv -1 \pmod 4$, $U_6(q)$, $q\equiv 1 \pmod 4$	+ − −
Even Characteristic	$L_2(16)$, $L_3(4)$ $U_3(16)$	+ −
Alternating Groups	A_8, A_9, A_{10}, A_{11}	+
Sporadic Groups	J_2, J_3, M_{22}, M_{23}, M^c, L H-S, C_3	+ −

At the present time, the structure of all groups G having such a Sylow 2-subgroup $S=S_1\times S_2$ with S_1, S_2 dihedral has been determined [**33**]. In particular, it has been shown that G is not simple. Using the methods of [**33**], F. Smith of Ohio State University is investigating the case that S_1 is dihedral or quasi-dihedral and S_2 is quasi-dihedral.[3]

The procedure for characterizing each of the known simple groups or families of groups of 2-rank 3 or 4 by their Sylow 2-subgroups is very uniform in conceptual outline, although somewhat variable in technical detail. In this paper we shall illustrate the entire process by presenting such a characterization of the families $G_2(q)$ and $D_4^2(q)$, q odd (which have isomorphic Sylow 2-subgroups for a given value of q). We note that for $q\equiv 3, 5 \pmod 8$, the Sylow 2-subgroups of these groups are also isomorphic to those of M_{12}.

[3] ADDED IN PROOF. This work has now been completed.

Each such characterization theorem rests ultimately on a preliminary characterization theorem, given in terms of the structure of the centralizer $M=C_G(z)$ of a central involution z of G (i.e., an involution z in the center of a Sylow 2-subgroup S of G). This preliminary theorem asserts that if M is isomorphic to the centralizer M^* of a central involution z^* of one of the known simple groups G^* of 2-rank 3 or 4 (or, more generally, of an extension G^* of such a simple group by a group of outer automorphisms of odd order), then G is, in fact, isomorphic to G^*. (In certain special cases, such as A_8 and A_9, there is more than one choice for G^*.) In the case of the groups $G_2(q)$, $D_4^2(q)$, q odd, the required result has been obtained by Harris [**17**], extending a prior characterization of these groups established by Fong and Wong [**14**], [**15**].

Thus, in effect, the entire aim of the analysis is to show, in our abstract simple group G with specified Sylow 2-subgroup S, that the structure of $M=C_G(z)$ is the same as that of the groups G^* we are trying to characterize. After an initial study of the possible fusion patterns for the involutions of G and *using prior characterization theorems to obtain the structure of* $M/\langle z\rangle$, one is able to show that

$$M/O(M) \cong M^*/O(M^*).$$

Hence the bulk of our work involves a proof that $O(M)=O(M^*)$. For most, but not all, of the known simple groups of 2-rank 3 or 4 and, in particular, for the groups $G_2(q)$, $D_4^2(q)$, q odd, one has that $O(M^*)=1$ and consequently in these cases our task is reduced to demonstrating that $O(M)=1$.

To accomplish this, one must show that if G is a minimal counterexample to the given classification theorem, then, in fact, G is a *balanced* group—that is, for any pair of commuting involutions a, b of G,

$$O(C_G(a)) \cap C_G(b) \subseteq O(C_G(b)).$$

(In the case that $O(M^*)\neq 1$, other variations of balance must be used. See [**31**], [**33**], [**34**].) At this point one is able to invoke Goldschmidt's improved version [**48**], [**49**] of the so-called "signalizer functor" theorem [**50**] together with some form of the so-called "balanced" theorem [**53**], [**54**] to show that G possesses a strongly embedded subgroup. Bender's theorem [**40**], [**41**] then yields a contradiction.

The proof of balance is itself fairly complicated. For the groups of 2-rank 3 and 4, it involves the prior construction of what we call "covering p-local subgroups," a notion which was first used effectively in the classification of groups with quasi-dihedral and wreathed Sylow 2-subgroups [**55**].

There is some evidence that in classification problems concerning simple groups of 2-rank at least 5 one may be able to get by without having to construct such covering p-local subgroups. If this turns out to be the case, it would provide a justification, if one feels that such is needed, for considering groups of 2-ranks 3 and 4 independently.

We remark also that since one proceeds inductively in any of these classification theorems, it is preferable not to demand, in practice, that G be simple, but only that G be *fusion-simple* (i.e., $O(G)=Z(G)=1$ and $O^2(G)=G$). Thus, in fact, it is necessary to classify all fusion-simple groups with the specified type of Sylow 2-subgroups. For example, the nonsplit extension $GL(3, 2)\cdot E_8^{(2)}$ of $GL(3, 2)$ by an elementary abelian group of order 8 is a fusion-simple, but nonsimple group with the same Sylow 2-subgroup as M_{12}.

Finally a word about problem A—the determination of the 2-groups which can occur as Sylow 2-subgroups of fusion-simple groups G of 2-rank 3 or 4. This problem has a natural subdivision; namely,

(I) The 2-local subgroups of G are all 2-constrained (in particular, solvable).

(II) Some 2-local subgroup of G is not 2-constrained.

The methods of Thompson's N-group paper [**45**], particularly §§13, 14, 15, appear to be applicable to problem (I), which seems to be the more difficult of the two.

At the present time, we are investigating problem A in the special case that G has *sectional* 2-rank at most 4; that is, every section of G has 2-rank at most 4; equivalently, G involves no elementary abelian 2-groups of order 32. The advantage of considering this special case is that the condition "sectional 2-rank at most 4" carries over to homomorphic images as well as to subgroups and so is inductive, whereas the more general condition "2-rank at most 4" is not inductive. We note that, by a result of MacWilliams [**44**], this special case does include all groups in which $SCN_3(2)$ is empty; that is, in which a Sylow 2-subgroup possesses no elementary abelian normal subgroups of order 8.

2. **Groups of type** $G_2(q)$, q **odd.** As is customary, a group G is said to have Sylow 2-subgroups of *type* X if a Sylow 2-subgroup of G is isomorphic to that of the group X.

In the balance of this paper, we shall classify all fusion-simple groups and, in particular, all simple groups with Sylow 2-subgroups of type $G_2(q)$, q odd. A Sylow 2-subgroup S of $G_2(q)$, q odd, can be defined by generators a, b, t, u, subject to the relations:

$$
(1)\qquad \begin{gathered}[a, b] = [t, u] = 1, \quad a^t = b, \quad a^u = a^{-1}, \quad b^u = b^{-1}, \\ a^{2^n} = b^{2^n} = t^2 = u^2 = 1,\end{gathered}
$$

for some integer $n \geqq 2$.

Our main result is the following:

THEOREM A. *If G is a perfect fusion-simple group with Sylow 2-subgroups of type $G_2(q)$, q odd, then G is isomorphic to one of the following groups*:

$$GL(3, 2)\cdot E_8^{(2)}, \quad M_{12}, \quad G_2(r), \quad \text{or} \quad {}^2D_4(r) \quad \text{for some odd } r.$$

Using Harris' theorem [17], mentioned in the preceding section, we actually derive Theorem A as a corollary of the following result:

THEOREM B. *If G is a fusion-simple group with Sylow 2-subgroups of type $G_2(q)$, q odd, then one of the following holds*:

(i) *G has two conjugacy classes of involutions and*

$$G \cong M_{12} \quad \text{or} \quad GL(3, 2)\cdot E_8^{(2)};$$

(ii) *G has one conjugacy class of involutions and if $M = C_G(z)$, z an involution of G, then*

(a) *$O(M) = 1$;*

(b) *$M' \cong SL(2, q_1) * SL(2, q_2)$ for suitable odd q_1, q_2;*

(c) *M/M' has Sylow 2-subgroups of order 2.*

We remark that our proof of Theorem B is independent of Harris' theorem, which is thus needed solely to derive Theorem A from Theorem B.

Our notation will be standard and will include the use of the "bar" convention for homomorphic images.

3. Balance, p-stability, and Harris' theorem. In this section we describe the three results which will be basic for the proof of Theorems A and B.

First, it is immediate from the definitions that in a balanced group G, O is an A-signalizer functor on G for every elementary abelian 2-subgroup A of G of rank at least 3. It follows therefore from Goldschmidt's fundamental result concerning A-signalizer functors that the group

$$\langle O(C_G(a)) \mid a \in A^\#\rangle$$

is of odd order.

Because of this result, the "2-generated" case of the balanced theorem of [53], [54] now holds in balanced groups of 2-rank at least 3. We recall that a group H is said to be *2-generated* if H is generated by its subgroups $N_H(Q)$, where Q ranges over the subgroups of 2-rank at least 2 in a fixed Sylow 2-subgroup of H. Likewise we recall that a group G is *connected* if for every pair of noncyclic elementary abelian 2-subgroups A, A' of a fixed Sylow 2-subgroup S of G, there exists a chain of noncyclic elementary abelian 2-subgroups

$$A = A_1, A_2, \cdots, A_n = A'$$

of S such that either

$$A_i \subseteq A_{i+1} \quad \text{or} \quad A_{i+1} \subseteq A_i \qquad (1 \leqq i \leqq n-1).$$

Goldschmidt's theorem together with the argument of §§4.2 and 4.3 of [53] yields the following result:

THEOREM 3.1. *If G is a balanced, connected group of 2-rank at least 3 with $O(G)=1$ in which the centralizer of every involution is 2-generated, then $O(C_G(x))=1$ for every involution x of G.*

We also recall from [55] that a group H with $O_p(H)\neq 1$, p odd, is said to be *p-stable with respect to* the p-subgroup P of H provided

(a) $P \cap O_{p',p}(H)$ is a Sylow p-subgroup of $O_{p',p}(H)$;

(b) Either P is normal in a Sylow p-subgroup of H or PK/K contains $O_p(H/K)$ for every normal subgroup K or H; and

(c) For any nontrivial normal subgroup P_0 of P such that $O_{p'}(H)P_0$ is normal in H, we have

$$AC_H(P_0)/C_H(P_0) \subseteq O_p(N_H(P_0)/C_H(P_0))$$

for every subgroup A of P such that $[P_0, A, A]=1$.

We can now state the extended form of Glauberman's ZJ-Theorem [55, Theorem 2.7.2].

THEOREM 3.2. *If H is a group with $O_p(H)\neq 1$, p odd, which is p-constrained and is p-stable with respect to the p-subgroup P of H, then*

$$H = O_{p'}(H)N_H(Z(J(P))).$$

Theorem 3.2 is an essential tool in the construction of covering p-local subgroups.

Finally we state Harris' theorem.

THEOREM 3.3. *Let G be a fusion-simple group with Sylow 2-subgroups of type $G_2(q)$, q odd. If z is an involution in the center of a Sylow 2-subgroup of G and if $M=C_G(z)$ has the structure specified in Theorem*

B(ii), *then* G *possesses a normal subgroup of odd index isomorphic to* $G_2(r)$ *or* $D_4^2(r)$ *for some odd* r. *In particular, if* G *is perfect, then* $G \cong G_2(r)$ *or* $D_4^2(r)$.

We shall also need a fundamental characterization of M_{12}, obtained by Brauer and Fong [**29**]. However, we prefer to state this result in the next section as part of our discussion of the fusion of involutions of G.

4. **The involutions of** G. We let G be a fusion-simple group with Sylow 2-subgroup S defined by the generators and relations (1) and we fix this notation for the paper.

Our object in this section will be to determine the fusion pattern of involutions of G and to obtain partial information concerning the structure of the centralizers of the involutions of G.

We introduce some additional notation. We set

$$z_1 = a^{2^{n-1}}, \qquad z_2 = b^{2^{n-1}}, \qquad z = z_1 z_2, \qquad Z = \langle z_1, z_2\rangle,$$
$$W = \langle a\rangle \times \langle b\rangle, \qquad Q_1 = \langle ab^{-1}, z_1 t\rangle, \qquad Q_2 = \langle ab, z_1 tu\rangle,$$
$$R = Q_1 Q_2, \qquad T = \langle W, u\rangle, \qquad V_1 = \langle W, t\rangle, \quad \text{and} \quad V_2 = \langle W, tu\rangle.$$

The various parts of the following omnibus lemma are easy consequences of the definitions of the specified subgroups of S.

LEMMA 4.1. *The following conditions hold*:

(i) $Z(S) = \langle z\rangle$.

(ii) $\Omega_1(S) = S$, $\Omega_1(S') = \langle z_1, z_2\rangle$.

(iii) S *is connected of* 2-*rank* 3.

(iv) *The centralizer of every involution of* S *has* 2-*rank* 3.

(v) *Let* A *be an elementary abelian* 2-*subgroup of* S *of order* 8. *Then* $C_S(A) = A$ *and if* $Z \subset A \subset R$, *then* $N_S(A)/A \cong D_8$.

(vi) $\mathrm{Aut}(S)$ *is a* 2-*group and* $\mathrm{Aut}(R)$ *is a* $\{2, 3\}$-*group*.

(vii) R *is the central product of* Q_1 *and* Q_2, *each of which is isomorphic to* $Q_{2^{n+1}}$. *Moreover*, $R = Q_1 Q_2$ *is the unique representation of* R *as a central product of generalized quaternion subgroups*.

(viii) $W \cong Z_{2^n} \times Z_{2^n}$ *and* $\Omega_1(W) = Z$.

(ix) $C_S(Z) = T = \langle W, u\rangle$ *and* u *inverts* W.

(x) T *contains* $\frac{1}{2}|T|$ *elementary abelian subgroups of order* 8.

(xi) T, V_1, V_2 *are the maximal subgroups of* S *containing* W *and* $V_1 \cong V_2 \cong Z_{2^n} \int Z_2$.

(xii) $Q_i\langle ub\rangle$ *is quasi-dihedral of order* 2^{n+2}, $i = 1, 2$.

The following results, respectively, by Brauer-Fong [**29**] and Fong (unpublished) will be important to us.[4]

THEOREM 4.2. *If $n=2$, then the following conditions hold:*
(i) *G has one or two conjugacy classes of involutions.*
(ii) *If G has two conjugacy classes of involutions, then $G \cong M_{12}$ or $GL(3, 2) \cdot E_8^{(2)}$.*

THEOREM 4.3. *If $n \geqq 3$, then G has only one conjugacy class of involutions.*

In the course of the proof of Theorem 4.3, Fong has also obtained the following additional two results:

LEMMA 4.4. *If X is a finite group with Sylow 2-subgroup S and X has no isolated involution, then*

$$N_X(Z)/C_X(Z) \cong S_3.$$

LEMMA 4.5. *If G has only one conjugacy class of involutions, then*
(i) *All involutions of $R - \langle z \rangle$ are conjugate in $C_G(z)$.*
(ii) *$C_G(z)$ has a normal subgroup of index 2 with Sylow 2-subgroup R.*

In view of Theorems 4.2 and 4.3, Theorems A and B hold if G has more than one conjugacy class of involutions. We can therefore assume henceforth that *G has only one conjugacy class of involutions.* We set $M = C_G(z)$ and $\overline{M} = M/O(M)$ and fix this notation as well.

PROPOSITION 4.6. *We have*

$$\overline{M}' = \overline{L}_1 \times \overline{L}_2 \cong SL(2, q_1) * SL(2, q_2),$$

with q_i odd and $\overline{Q}_i$ a Sylow 2-subgroup of $\overline{L}_i$, $i = 1, 2$.

PROOF. By Lemma 4.5(ii), M has a normal subgroup K of index 2 with Sylow 2-subgroup R. We have $M' \subseteq K$. On the other hand, the focal subgroup $R_1 = \langle x^{-1}x^m \mid x \in S,\ x^m \in S,\ m \in M \rangle$ is a Sylow 2-subgroup of M'. However, by Lemma 4.5(i) and the structure of R, it follows that $R \subseteq R_1$. Hence $R = R_1$ and we conclude that $O^2(M) = K$. Thus $O^2(K) = K$.

Now $R = Q_1Q_2 \cong Q_{2^{n+1}} * Q_{2^{n+1}}$, by Lemma 4.1(vii), and so $R/\langle z \rangle \cong D_{2^n} \times D_{2^n}$. Setting $\tilde{M} = \overline{M}/\langle \bar{z} \rangle$, we have that $\tilde{R} = \tilde{Q}_1 \times \tilde{Q}_2$ is a Sylow 2-subgroup of $\tilde{K}$ and $\tilde{R} \cong D_{2^n} \times D_{2^n}$. Furthermore, our conditions imply that $O^2(\tilde{K}) = \tilde{K}$ and $O(\tilde{K}) = 1$. If $n > 2$, we apply the main result of

[4] Because Fong's results are unpublished, we shall for completeness present a proof of Theorem 4.3 and Lemmas 4.4 and 4.5 in an appendix to this paper.

[**33**] to obtain that $\tilde{R}\subseteq\tilde{K}'$ and that $\tilde{K}'=\tilde{L}_1\times\tilde{L}_2$, where $\tilde{L}_i\cong PSL(2, q_i)$ or A_7, $i=1, 2$. If $\overline{L}_i$ denotes the inverse image of $\tilde{L}_i$ in $\overline{K}$, then by the structure of $\overline{R}$, $\overline{L}_i$ does not split and so by the results of Schur $\overline{L}_i\cong SL(2, q_i)$ or $\hat{A}_7$, $i=1, 2$, where, as usual, $\hat{A}_7$ denotes the nonsplit extension of A_7 by Z_2. Since $\overline{R}$ is a Sylow 2-subgroup of $\overline{L}_1\overline{L}_2$, it follows that $\overline{R}$ is the central product of $\overline{R}\cap\overline{L}_1$ and $\overline{R}\cap\overline{L}_2$, each of which is generalized quaternion. We conclude therefore from Lemma 4.1(vii), for a suitable choice of the numbering of $\overline{L}_1$ and $\overline{L}_2$, that $\overline{Q}_i=\overline{R}\cap\overline{L}_i$ and hence that $\overline{Q}_i$ is a Sylow 2-subgroup of $\overline{L}_i$, $i=1, 2$.

Since ub leaves Q_1 and Q_2 invariant, $\bar{u}\bar{b}$ leaves $\overline{L}_1$ and $\overline{L}_2$ invariant, whence also $\tilde{u}\tilde{b}$ leaves $\tilde{L}_1$ and $\tilde{L}_2$ invariant. Hence if $\tilde{L}_i\cong A_7$, it would follow that $\tilde{L}_i\langle\tilde{u}\tilde{b}\rangle\cong S_7$ or $A_7\times Z_2$. However, by Lemma 4.1(xii), the Sylow 2-subgroup $\tilde{Q}_i\langle\tilde{u}\tilde{b}\rangle$ of $\tilde{L}_i\langle\tilde{u}\tilde{b}\rangle$ is dihedral of order 16 (as $n=3$ in this case). Since the Sylow 2-subgroups of S_7 and $A_7\times Z_2$ are not dihedral, we reach a contradiction and so $\overline{L}_i\cong SL(2, q_i)$, $i=1, 2$. Furthermore, we have that $\tilde{M}=\tilde{K}\langle\tilde{u}\tilde{b}\rangle$ with $\tilde{K}/\tilde{L}_1\tilde{L}_2$ abelian of odd order. Since $\tilde{u}\tilde{b}$ leaves each $\tilde{L}_i$ invariant, it follows directly from the structure of $P\Gamma L(2, q_i)$ that $\tilde{M}/\tilde{L}_1\tilde{L}_2$ is abelian, whence also $\overline{M}/\overline{L}_1\overline{L}_2$ is abelian Thus $\overline{M}'=\overline{L}_1\overline{L}_2$ and so all parts of the lemma hold in this case.

Suppose next that $n=2$. In this case we apply the main result of [**39**] to obtain that $\tilde{R}\subseteq\tilde{K}'$ and that either $\tilde{K}\cong PSL(2,16)$ or $\tilde{K}'=\tilde{F}_1\times\tilde{F}_2$, where $\tilde{F}_i\cong PSL(2, q_i)$, $q_i\equiv 3, 5 \pmod 8$, $q_i\geqq 5$, or $\tilde{F}_i\cong Z_2\times Z_2$, $i=1, 2$. Now $\tilde{K}$ does not possess a 5-element which normalizes, but does not centralize $\tilde{R}$, since otherwise $\mathrm{Aut}(\overline{R})$ and hence also $\mathrm{Aut}(R)$ would be divisible by 5, contrary to Lemma 4.1(vi). In particular, it follows that $\tilde{K}$ is not isomorphic to $PSL(2, 16)$ and, in addition, if $\tilde{K}'$ is elementary of order 16, then $\tilde{K}/\tilde{K}'$ is a 3-group. Moreover, in the latter case, Lemma 4.5(i) implies that $\tilde{K}/\tilde{K}'$ is elementary of order 9, whence $\tilde{K}$ is of the form $\tilde{L}_1\times\tilde{L}_2$, where $\tilde{L}_i\cong PSL(2, 3)$, $i=1, 2$. Letting $\overline{L}_i$ denote the inverse image of $\tilde{L}_i$ in $\overline{M}$, we see exactly as in the case of $n>2$ that all parts of (ii) hold. Likewise if $\tilde{F}_i\cong PSL(2, q_i)$ for both $i=1$ and 2, we let $\overline{L}_i$ denote the inverse image of $\tilde{F}_i$ in $\overline{M}$ and reach the same conclusion.

Suppose finally that $\tilde{F}_i=\tilde{Q}_i\cong Z_2\times Z_2$ and $\tilde{F}_j\cong PSL(2, q_j)$, $i\neq j$, say $i=1$, $j=2$. Since $O^2(\tilde{K})=\tilde{K}$, $\tilde{K}$ contains a 3-element $\tilde{x}$, which normalizes, but does not centralize $\tilde{Q}_1$. By the Frattini argument, we can take $\tilde{x}$ to normalize $\tilde{R}$. But by the structure of $\tilde{M}$, we have that $C_{\tilde{M}}(\tilde{R})$ has a normal 2-complement and that $N_{\tilde{M}}(\tilde{R})/C_{\tilde{M}}(\tilde{R})$ has an elementary abelian normal subgroup of order 9 and index 2. Since $\tilde{u}\tilde{b}$ normalizes $\tilde{R}$ and does not centralize $\tilde{Q}_1$, we see that we can, in fact, choose $\tilde{x}$ to be inverted by some element $\tilde{v}$ of $\tilde{S}-\tilde{R}$. How ever, $P\Gamma L(2,q_2)/PSL(2,q_2)$ is abelian and $\langle\tilde{x},\tilde{v}\rangle\cong S_3$ normalizes

$\tilde{F}_2 \cong PSL(2, q_2)$. Hence $\tilde{x} \in C_{\tilde{M}}(\tilde{F}_2) \times \tilde{F}_2$ and so $C_{\tilde{M}}(\tilde{F}_2) \supset \tilde{Q}_1$. Since $\tilde{Q}_1 \cong Z_2 \times Z_2$, we conclude that $C_{\tilde{M}}(\tilde{F}_2) \cong A_4 \cong PSL(2, 3)$. Letting $\overline{L}_1$, $\overline{L}_2$ be the inverse images of $C_{\tilde{M}}(\tilde{F}_2)$ and $\tilde{F}_2$ in $\overline{M}$ respectively, it follows as in the preceding cases that the lemma holds.

In view of the proposition, we see that Theorem B will be established once we prove that $O(M)=1$. This we shall do in the balance of the paper.

The integers q_1, q_2 of the proposition are uniquely determined by G inasmuch as G has only one conjugacy class of involutions. We call them the *characteristic powers* of G.

As a corollary of the proposition, we obtain the following additional properties of $\overline{M}$:

LEMMA 4.7. *The following conditions hold*:

(i) $\overline{M} = \overline{L}_1\overline{L}_2 C_{\overline{M}}(\overline{S})\langle \bar{u}\bar{b}\rangle$.

(ii) *If $\overline{A}$ is an elementary abelian subgroup of $\overline{R}$ of order* 8, *then* $O(C_{\overline{M}}(\overline{A})) = O(C_{\overline{M}}(\overline{S}))$.

(iii) *$C_{\overline{M}}(\overline{Z})$ has a normal* 2-*complement which is the unique maximal $\overline{T}$-invariant subgroup of $\overline{M}$ of odd order.*

PROOF. By Lemma 4.1(vi), $\mathrm{Aut}(S)$ is a 2-group. Moreover, we have already shown in the preceding proof that $\overline{M}/\overline{L}_1\overline{L}_2$ is abelian. Hence $\overline{L}_1\overline{L}_2\overline{S} \lhd \overline{M}$ and so by the Frattini argument, $\overline{M} = \overline{L}_1\overline{L}_2 N_{\overline{M}}(\overline{S}) = \overline{L}_1\overline{L}_2 C_{\overline{M}}(\overline{S})\overline{S}$. Since $\overline{S} = \overline{R}\langle \bar{u}\bar{b}\rangle$ and $\overline{R} \subseteq \overline{L}_1\overline{L}_2$, we see that (i) holds.

Again with $\tilde{M} = \overline{M}/\langle \bar{z}\rangle$ and with $\overline{A}$ as in (ii), we have that $\tilde{A}$ is a "diagonal" four-subgroup of $\tilde{L}_1\tilde{L}_2$; that is, the image of $\tilde{A}$ in $\tilde{L}_1\tilde{L}_2/\tilde{L}_i$ is a four-group for $i=1, 2$. Since $C_{\tilde{L}_i}(\tilde{W}_i) = \tilde{W}_i$ for any four-subgroup $\tilde{W}_i$ of $\tilde{L}_i$, $i=1, 2$, we conclude that $C_{\tilde{L}_1\tilde{L}_2}(\tilde{A}) = \tilde{A}$, whence $C_{\overline{L}_1\overline{L}_2}(\overline{A}) = \overline{A}$. Since $C_{\overline{M}}(\overline{S})$ has a normal 2-complement which centralizes $\overline{A}$, (ii) now follows at once from (i).

Similarly the image of $\tilde{Z} = \langle \tilde{z}_1, \tilde{z}_2\rangle$ in $\tilde{L}_1\tilde{L}_2/\tilde{L}_i$ is an involution for $i=1, 2$. Since $C_{\tilde{L}_i}(\tilde{v}_i)$ has a normal 2-complement for any involution $\tilde{v}_i$ of $\tilde{L}_i$, we see that $C_{\tilde{L}_i}(\tilde{Z})$ has a normal 2-complement and consequently $\overline{D}_i = O(C_{\overline{L}_i}(\overline{Z}))$ is a normal 2-complement in $C_{\overline{L}_i}(\overline{Z})$, $i=1, 2$. Moreover, $\overline{D}_i$ is invariant under $(\overline{T} \cap \overline{Q}_i)\langle \bar{u}\bar{b}\rangle$ and $\overline{Q}_i\langle \bar{u}\bar{b}\rangle$ is a Sylow 2-subgroup of $\overline{L}_i\langle \bar{u}\bar{b}\rangle$ with $\overline{Q}_i\langle \bar{u}\bar{b}\rangle$ being quasi-dihedral, $i=1, 2$. It follows now from the structure of $\overline{L}_i\langle \bar{u}\bar{b}\rangle$ that $\overline{D}_i$ is, in fact, the unique maximal $(\overline{T} \cap \overline{Q}_i)\langle \bar{u}\bar{b}\rangle$-invariant subgroup of $\overline{L}_i$ of odd order. We therefore conclude that $\overline{D}_1\overline{D}_2 = O(C_{\overline{L}_1\overline{L}_2}(\overline{Z}))$ and that $\overline{D}_1\overline{D}_2$ is the unique maximal $\overline{T}$-invariant subgroup of $\overline{L}_1\overline{L}_2$ of odd order. Clearly then by (i) we have that $\overline{D}_1\overline{D}_2 O(C_{\overline{M}}(\overline{S}))$ is a normal 2-complement in $C_{\overline{M}}(\overline{Z})$ and is the unique maximal $\overline{T}$-invariant subgroup of $\overline{M}$ of odd order, proving (iii).

Finally we prove

LEMMA 4.8. *The centralizer of every involution of G is 2-generated.*

PROOF. Since G has only one conjugacy class of involutions, we need only prove that $X=\Gamma_{S,2}(M)=M$. Since S has 2-rank 3, certainly $O(M)N_M(S)\subseteq X$. Hence, by Lemma 4.7(i), it will suffice to show that $\bar{L}_1\bar{L}_2\subseteq\bar{X}=\Gamma_{\bar{S},2}(\bar{M})$.

Let x_i be arbitrary elements of order 4 in Q_i, $i=1, 2$. Then $U=\langle x_1x_2, z\rangle$ is a four-subgroup of S and so, for $i=1, 2$, $C_{\bar{L}_i}(\bar{x}_i)=C_{\bar{L}_i}(\bar{U})\subseteq\bar{X}$. But by the structure of $SL(2, q_i)$, one checks that $\bar{L}_i$ is generated by its subgroups $C_{\bar{L}_i}(\bar{x}_i)$ as $\bar{x}_i$ ranges over the elements of order 4 in the Sylow 2-subgroup $\bar{Q}_i$ of $\bar{L}_i$ except in the case $q_i=3$ or 5. Thus if $q_i>5$, we conclude that $\bar{L}_i\subseteq\bar{X}$. However, if $q_i=3$, then $\bar{L}_i\subseteq N_{\bar{M}}(\bar{R})\subseteq\bar{X}$. On the other hand, if $q_i=5$, we use the four-subgroup $U_1=\langle ub,z\rangle$ and check that $\bar{L}_i=\langle N_{\bar{L}_i}(\bar{Q}_i), C_{\bar{L}_i}(\bar{u}\bar{b})\rangle=\langle N_{\bar{L}_i}(\bar{R}), C_{\bar{L}_i}(\bar{U}_1)\rangle\subseteq\bar{X}$. Hence $\bar{L}_1\bar{L}_2\subseteq\bar{X}$ in all cases and the proposition is proved.

5. **Subgroup structure of** G. By Theorem 3.3, Theorem B implies Theorem A. Thus, in proving these theorems, we can assume henceforth that G is a minimal counterexample to Theorem B. Proposition 4.6 then yields that $O(M)\neq 1$.

In this section we establish sufficient information concerning the subgroup structure of G to allow us to construct covering p-local subgroups of G for the primes p dividing $|O(M)|$.

We set $N=N_G(Z)$ and $C=C_G(Z)$ and fix this notation. We first prove

LEMMA 5.1. *The following conditions hold:*

(i) $N/C\cong S_3$.

(ii) $T=\langle u, W\rangle$ *is a Sylow 2-subgroup of C and C has a normal 2-complement.*

PROOF. First, (i) is a restatement of Lemma 4.4. Since $Z\lhd S$, $S\subseteq N$. Since $C\lhd N$, $S\cap C=C_S(Z)=C_S(z_2)$ is thus a Sylow 2-subgroup of C. Since $C_S(z_2)=T$, we conclude that T is a Sylow 2-subgroup of C.

We argue now that C has a normal subgroup K of index 2 with Sylow 2-subgroup W. Since $W=\langle a, b\rangle\cong Z_{2^n}\times Z_{2^n}$ and $Z=\Omega_1(W)$, a theorem of Brauer [**57**] will yield that K, and hence also C, has a normal 2-complement.

Since $Z=\Omega_1(W)$ and $C=C_G(Z)$, obviously no involution of $T-W$ can be conjugate in C to an involution of W. It follows therefore from Thompson's fusion lemma that C possesses a subgroup K of index 2 not containing any element of $T-W$. Thus W is a Sylow 2-subgroup of K and the proof is complete.

LEMMA 5.2. *If A is any elementary abelian subgroup of R of order* 8 *containing Z, then we have*

(i) $C_G(A)=A\times O(C_G(A))$.

(ii) $N_G(A)/C_G(A)\cong GL(3, 2)$.

PROOF. By Lemma 4.1(v), every elementary abelian subgroup of S of order 8 is self-centralizing in S. This implies that A is necessarily a Sylow 2-subgroup of $C_G(A)$ and so (i) follows from Burnside's transfer theorem.

Since $R\cong Q_{2^{n+1}} * Q_{2^{n+1}}$, $A\subseteq R_1\subseteq R$, where $R_1\cong Q_8 * Q_8$. Furthermore, by the structure of $\overline{M}$ given in Proposition 4.6, it follows that $N_G(R_1)/C_G(R_1)$ contains an elementary subgroup of order 9. Since R_1 contains exactly 6 elementary abelian groups of order 8, we conclude that $|N_M(A)/C_M(A)|$ is divisible by 3. Furthermore, as $A\supseteq Z$, Lemma 4.1(v) also yields that $N_S(A)/A\cong D_8$. Thus $N_M(A)/C_M(A)\cong S_4$.

To prove (ii), it will thus clearly suffice to show that $N_G(A)\not\subseteq M$. Set $\overline{N}=N/O(N)$. By the preceding lemma, $\overline{T}\lhd\overline{N}$, $\overline{T}=C_{\overline{N}}(\overline{Z})$, and $\overline{N}/\overline{T}\cong S_3$. Then $\overline{N}$ contains an element $\bar{x}$ of order 3 and $\bar{x}$ does not centralize $\overline{Z}$. Since $\overline{Z}=\Omega_1(\overline{W})$ with $\overline{W}\cong Z_{2^n}\times Z_{2^n}$ and with the elements of $\overline{T}-\overline{W}$ inverting $\overline{W}$, we must have $[\overline{W}, \bar{x}]=\overline{W}$ and also $C_{\overline{T}}(\bar{x})$ of order 2. Clearly $\bar{x}$ normalizes $\overline{Z}C_{\overline{T}}(\bar{x})\cong E_8$. On the other hand, if $\overline{B}$ is any elementary subgroup of $\overline{T}$ of order 8 normalized by $\bar{x}$, we have $\overline{B}=\overline{Z}C_{\overline{B}}(\bar{x})=\overline{Z}C_{\overline{T}}(\bar{x})$, so $\bar{x}$ normalizes a unique elementary abelian subgroup of $\overline{T}$ of order 8. But $\overline{T}\langle\bar{x}\rangle$ contains exactly $|\overline{T}|/2$ Sylow 3-subgroups as $|C_{\overline{T}}(\bar{x})|=2$. However, by Lemma 4.1(x), $\overline{T}$ also contains exactly $|\overline{T}|/2$ elementary subgroups of order 8. Thus each is normalized by an element of order 3. But $A\subseteq T$ as $A\supseteq Z$. We see then that $N_G(A)$ contains a 3-element which normalizes, but does not centralize Z. Hence $N_G(A)\not\subseteq M$ and (ii) is proved.

We next prove

PROPOSITION 5.3. *If H is a proper subgroup of G containing S, and we set $\overline{H}=H/O(H)$, then one of the following holds*:

(i) $H=O(H)(H\cap M)$.

(ii) $H=O(H)(H\cap N)$ *and* $H/O(H)T\cong S_3$.

(iii) $O^2(\overline{H})$ *contains a normal subgroup of odd index isomorphic to* $PSL(3, q)$, $q\equiv 1 \pmod 4$ *or* $PSU(3, q)$, $q\equiv -1 \pmod 4$, *and with Sylow 2-subgroup* $\overline{V}_1$ *or* $\overline{V}_2$.

(iv) $\overline{H}$ *is fusion-simple. More precisely, either $\overline{H}$ contains a simple normal subgroup of odd index or* $\overline{H}\cong GL(3, 2)\cdot E_8^{(2)}$.

PROOF. If z is an isolated involution of H, then $H=O(H)C_H(z) = O(H)(H\cap M)$ by Glauberman's Z^*-theorem and (i) holds. Hence we may assume that $Z^*(\overline{H})=1$.

Suppose next that $O^2(\overline{H})=\overline{H}$, in which case $\overline{H}$ is fusion-simple. Since $\overline{H}$ has Sylow 2-subgroups of type $G_2(q)$, q odd, and since $|\overline{H}|<|G|$, Theorem B holds for $\overline{H}$ by our minimal choice of G. If $\overline{H}\cong M_{12}$ or $GL(3, 2)\cdot E_8^{(2)}$, then (iv) holds. In the contrary case, $\overline{H}$ has only one conjugacy class of involutions. Let $\overline{L}$ be a minimal normal subgroup of $\overline{H}$. Since $O(\overline{H})=1$, $\overline{L}$ is of even order and as $\overline{H}$ has only one conjugacy class of involutions, all involutions of $\overline{H}$ thus lie in $\overline{L}$. But $S=\Omega_1(S)$ by Lemma 4.1(ii) and so $\overline{S}\subseteq\overline{L}$, whence $\overline{L}$ is of odd index in $\overline{H}$. Furthermore, the minimality of $\overline{L}$ implies that $\overline{L}$ is the direct product of isomorphic simple groups. Since $\overline{S}$ cannot be expressed as a direct product, we conclude that $\overline{L}$ is simple and so (iv) holds in this case.

Thus it remains to treat the case that $\overline{K}=O^2(\overline{H})\subset\overline{H}$. By Lemma 4.4, $N\cap H/C\cap H\cong N/C$. As noted in the preceding proof, $N_N(W)$ contains a 3-element x such that $[W, x]=W$. In view of the stated isomorphism, we can take x in H and consequently $\overline{W}\subseteq\overline{K}$. If $\overline{W}=\overline{K}\cap\overline{S}$, then Brauer's theorem [57] implies that $\overline{W}\lhd\overline{K}$, in which case $\overline{Z}$ is normal in both $\overline{K}$ and $\overline{H}$. We conclude at once, using Lemma 5.1, that (ii) holds. Therefore we may assume that $\overline{K}\cap\overline{S}\supset\overline{W}$, whence $\overline{K}\cap\overline{S}=\overline{T}$, $\overline{V}_1$, or $\overline{V}_2$ by Lemma 4.1(xi). The latter two groups are isomorphic to $Z_{2^n}\int Z_2$ by Lemma 4.1(xi). Since $\overline{K}$ is fusion-simple and $O(\overline{K})=1$, the main result of [56] implies in these cases that (iii) holds.

Suppose finally that $\overline{K}\cap\overline{S}=\overline{T}=\langle\overline{W}, \bar{u}\rangle$. We shall argue in this case that $\overline{K}$ has a normal subgroup of index 2, which will contradict the fact that $O^2(\overline{K})=\overline{K}$ and will thus complete the proof. It will clearly suffice to show that the inverse image K of $\overline{K}$ in H has a normal subgroup of index 2. We have that T is a Sylow 2-subgroup of K. Suppose that an involution v of $T-W$ is conjugate in K to an element of W and hence of $Z=Z(T)$. Then $C_K(v)$ contains a Sylow 2-subgroup T_1 of K, which without loss we may assume contains $C_T(v)$. By Lemma 4.1(ix), v inverts W and so $C_T(v)=\langle Z, v\rangle$ is elementary of order 8. Since T_1 has 2-rank 3, we must have $Z(T_1)\subseteq\langle Z, v\rangle$. Since $v\in Z(T_1)$, it follows that $Z(T_1)=\langle v, x\rangle$ for some x in Z. Since $W\cong Z_{2^n}\times Z_{2^n}$, there exists an element y in W such that $y^2=x$. Observe that

$$v^y = y^{-1}vyvv = y^{-2}v = xv.$$

Since $x^v=x$, we see that y normalizes, but does not centralize $\langle v, x\rangle = Z(T_1)$. This is a contradiction, since obviously $|N_K(Z(T_1))/C_K(Z(T_1))|$

is odd as T_1 is a Sylow 2-subgroup of K. Thus v is not conjugate to an element of W. Since W is a maximal subgroup of T, Thompson's fusion lemma now yields that K has a normal subgroup of index 2 (with Sylow 2-subgroup W).

REMARK. The final argument of the proposition actually yields the following general result, which we shall need in the Appendix. Let T^* be a 2-group of the form $W^*\langle u^*\rangle$, where W^* is the direct product of two cyclic groups, each of order at least 4, and u^* is an involution which inverts W^*. Then if X is any group with Sylow 2-subgroup T^*, our argument shows that X possesses a normal subgroup of index 2, with Sylow 2-subgroup W^*.

As a consequence of the proposition, we have

LEMMA 5.4. *Let H be a proper subgroup of G containing S and set $\overline{H}=H/O(H)$. If $\overline{D}$ is a maximal S-invariant subgroup of odd order, then $D\supseteq O(H)$ and $\overline{D}=O(C_{\overline{H}}(\overline{Z}))$. In particular, D is uniquely determined and Z centralizes $D/O(H)$.*

PROOF. We apply the preceding proposition and check each of the possibilities for $\overline{H}$. Clearly any maximal S-invariant subgroup of H of odd order contains $O(H)$; so it will suffice to prove that $\overline{D}=O(C_{\overline{H}}(\overline{Z}))$.

If $H=O(H)(H\cap M)$, then $\overline{H}=\overline{H\cap M}$ is isomorphic to a subgroup of $\overline{M}=M/O(M)$. However, we have shown in Lemma 4.7(iii) that $O(C_{\overline{M}}(\overline{Z}))$ is the unique maximal $\overline{T}$-invariant subgroup of odd order in $\overline{M}$. Since $O(C_{\overline{M}}(\overline{Z}))$ is, in fact, $\overline{S}$-invariant, we conclude at once that $\overline{D}=O(C_{\overline{H}}(\overline{Z}))$. If $H=O(H)(H\cap N)$ with $H/O(H)\cong S_3$, it follows from Lemma 5.1 that $\overline{D}=1$ and that $O(C_{\overline{H}}(\overline{Z}))=1$, so the lemma holds trivially.

Suppose next that $O^2(\overline{H})$ contains a normal subgroup $\overline{K}$ of odd index isomorphic to $PSL(3, q)$, $q\equiv 1 \pmod 4$ or $PSU(3, q)$, $q\equiv -1 \pmod 4$, with wreathed Sylow 2-subgroup $\overline{V}_1$ or $\overline{V}_2$, say, $\overline{V}_1$ for definiteness. We have that $\mathrm{Aut}(\overline{V}_1)$ is a 2-group. Furthermore, $\overline{H}/\overline{K}$ has a normal 2-complement as a Sylow 2-subgroup of $\overline{H}/\overline{K}$ is of order 2. Hence $\overline{H}=\overline{K}O(C_{\overline{H}}(\overline{V}_1))\overline{S}=\overline{K}O(C_{\overline{H}}(\overline{Z}))\overline{S}$ by the Frattini argument. Hence, to establish the lemma in this case, it will suffice to show that $O(C_{\overline{K}}(\overline{Z}))$ is the unique maximal $\overline{S}$-invariant subgroup of $\overline{K}$ of odd order. However, by the structure of $PSL(3, q)$, $q\equiv 1 \pmod 4$, and $PSU(3, q)$, $q\equiv -1 \pmod 4$, one checks that $O(C_{\overline{K}}(\overline{Z}))$ is, in fact, the unique maximal $\overline{V}_1$-invariant subgroup of $\overline{K}$ of odd order. Since $O(C_{\overline{K}}(\overline{Z}))$ is $\overline{S}$-invariant, the lemma therefore holds in this case.

Suppose finally that part (iv) of the proposition holds. If $\overline{H}\cong M_{12}$ or $GL(3, 2)\cdot E_8^{(2)}$, it follows at once from the structure of these groups that $\overline{D}=O(C_{\overline{H}}(\overline{Z}))=1$ and the lemma holds. In the contrary case, $\overline{H}$ contains a simple normal subgroup $\overline{K}$ of odd index not isomorphic to M_{12}. Our minimal choice of G implies that $\overline{K}$, and hence also $\overline{H}$, has only one conjugacy class of involutions and, moreover, that $\overline{M}_0=C_{\overline{H}}(\bar{z})$, $\overline{M}_1=C_{\overline{H}}(\bar{z}_1)$, and $\overline{M}_2=C_{\overline{H}}(\bar{z}_2)$ have the structure asserted in Theorem B, so that $O(\overline{M}_i)=1$ and $\overline{M}_i'\cong M'O(M)/O(M)$ for all i, $0\leqq i\leqq 2$.

Now $\overline{T}\subseteq\overline{M}_i$ for all i and so $\overline{T}\subseteq\overline{S}_i$ for some Sylow 2-subgroup $\overline{S}_i$ of $\overline{M}_i$, $i=1, 2$, and $\overline{Z}$, $\overline{T}$ play the same role in $\overline{S}_1$ and $\overline{S}_2$ as they do in $\overline{S}$. Hence Lemma 4.7(iii) applies as well to $\overline{M}_1$ and $\overline{M}_2$ as it does to $\overline{M}_0$ and yields that $\overline{D}_i=O(C_{\overline{M}_i}(\overline{Z}))$ is the unique maximal $\overline{T}$-invariant subgroup of odd order in $\overline{M}_i$, $0\leqq i\leqq 2$. Putting $z_0=z$, we have that $\overline{D}=\langle C_{\overline{D}}(\bar{z}_i)\,|\,0\leqq i\leqq 2\rangle$. Since $C_{\overline{D}}(\bar{z}_i)$ is a $\overline{T}$-invariant subgroup of $\overline{M}_i$ of odd order, $C_{\overline{D}}(\bar{z}_i)\subseteq\overline{D}_i$ for each i and we conclude that $\overline{D}\subseteq\langle\overline{D}_i\,|\,0\leqq i\leqq 2\rangle$. However, by Lemma 5.1, applied to $\overline{H}$, we have that $C_{\overline{H}}(\overline{Z})$ has a normal 2-complement, whence $\overline{D}_i=O(C_{\overline{M}_i}(\overline{Z}))\subseteq O(C_{\overline{H}}(\overline{Z}))$, $0\leqq i\leqq 2$, whence $\overline{D}\subseteq O(C_{\overline{H}}(\overline{Z}))$. The maximality of D now forces equality and the lemma holds in this case as well.

As a corollary of the lemma, we obtain a key transitivity theorem.

PROPOSITION 5.5. *Any two maximal S-invariant p-subgroups of G with a nontrivial intersection, p odd, are conjugate by an element of $C_G(S)$.*

PROOF. Suppose false and choose P_1, Q_1 maximal S-invariant p-subgroups of G with $P_1\cap Q_1\neq 1$ to violate the desired conclusion and such that $E=P_1\cap Q_1$ has maximal order. Setting $H=N_G(E)$, then H is a proper subgroup of G containing S and so by the preceding lemma, H possesses a unique maximal S-invariant subgroup D of odd order. Then $N_{P_1}(E)$ and $N_{Q_1}(D)$ are each contained in D.

We let P be a maximal S-invariant p-subgroup of G containing $N_{P_1}(E)$ such that $P\cap D$ is a Sylow p-subgroup D. Since $P\cap P_1\supseteq N_{P_1}(E)\supset E$, our maximal choice of P_1, Q_1 implies that $P_1=P^g$ for some g in $C_G(S)$. Furthermore, $N_{Q_1}(E)^h\subseteq P\cap D$ for some h in $C_D(S)\subseteq C_G(S)$. Then $P\cap Q_1^h\supseteq N_{Q_1}(E)^h\supset E$ and so $Q_1^{hg'}=P$ for some g' in $C_G(S)$, again by our maximal choice of P_1, Q_1. Setting $x=hg'g$, we have $x\in C_G(S)$ and $Q_1^x=P_1$, contrary to our choice of P_1, Q_1.

Finally we prove

LEMMA 5.6. *Let H be a p-local subgroup of G, p odd, with the following properties:*

(a) *H contains S and covers $M/O(M)$.*

(b) *H is p-constrained and $O_{p'}(H)\subseteq O(H)$.*

If P is a maximal S-invariant p-subgroup of H, then $H = O_{p'}(H)N_H(Z(J(P)))$.

PROOF. Except in a single case, we shall argue that H is p-stable with respect to P. The desired conclusion will then follow from the extended form of Glauberman's ZJ-theorem stated in Theorem 3.2. The exceptional case occurs when $H/O(H) \cong M/O(M)$, $p=3$, and one of the characteristic powers of G is 3 and the other is greater than 3. In this case, we shall show that $H = H_0N_H(P)$ for some subgroup H_0 of index 3 in H such that $O(H)PS \subseteq H_0$, H_0 is p-constrained, $O_{p'}(H_0) = O_{p'}(H)$, and such that H_0 is p-stable with respect to P. Glauberman's theorem will then yield $H_0 = O_{p'}(H_0)N_{H_0}(Z(J(P)))$. Since $H = H_0N_H(P)$, $O_{p'}(H_0) = O_{p'}(H)$, and $N_H(P)$ normalizes $Z(J(P))$, the desired conclusion $H = O_{p'}(H)N_H(Z(J(P)))$ will follow in this case as well.[5]

We first verify that P satisfies conditions (a) and (b) in the definition of p-stability with respect to P. Since $O_{p'}(H) \subseteq O(H)$, so also $O_{p',p}(H) \subseteq O(H)$ and consequently $O_{p',p}(H) = O_{p',p}(O(H))$. But by the maximality of P, $P \cap O(H)$ is a Sylow p-subgroup of $O(H)$ and hence $P \cap O_{p',p}(O(H))$ is a Sylow p-subgroup of $O_{p',p}(O(H))$. Thus, in fact, $P \cap O_{p',p}(H)$ is a Sylow p-subgroup of $O_{p',p}(H)$, and so condition (a) holds.

We claim that $\bar{H} = H/O(H)$ satisfies (i) or (iv) of Proposition 5.3. Indeed, by assumption (a) of the lemma, $C_{\bar{H}}(\bar{z}) \cong M/O(M)$ and so by Proposition 4.6 contains a normal subgroup isomorphic to $SL(2, q_1) * SL(2, q_2)$. But then Lemma 5.1 shows that $\bar{H}$ is not isomorphic to $N/O(N)$. Likewise using the structure of the centralizer of an involution in $PSL(3, q)$, $q \equiv 1 \pmod 4$ or $PSU(3, q)$, $q \equiv -1 \pmod 4$, we see that $\bar{H}$ cannot have the structure of Proposition 5.3(iii). Thus our assertion is proved.

Consider first the case that $\bar{H}$ contains a simple normal subgroup $\bar{L}$ of odd index. Since $\mathrm{Aut}(S)$ is a 2-group, we have $\bar{H} = \bar{L}C_{\bar{H}}(\bar{S}) = \bar{L}O(C_{\bar{H}}(\bar{S}))$ by the Frattini argument. Now let K be an arbitrary normal subgroup of H and set $\tilde{H} = H/K$. We argue that $\tilde{P} \supseteq O_p(\tilde{H})$. Suppose $K \subseteq O(H)$. Since $O(\bar{H}) = 1$, it follows that $O(H)$ covers $O_p(\tilde{H})$. Hence if E denotes the inverse image of $O_p(\tilde{H})$ in H, we have that $E \lhd H$ and $E \subseteq O(H)$. Since $P \cap O(H)$ is a Sylow p-subgroup of

[5] This exceptional case can be avoided if one first notes that the extended Glauberman ZJ-theorem [55, Theorem 2.7.2] and Theorem 2.2 above, actually holds with condition (b) in the definition of p-stability with respect to P replaced by the weaker condition:

(b′) *$N_H(P)K/K$ contains $O_p(H/K)$ for every normal subgroup K of H.*

$O(H)$, $P\cap E$ is a Sylow p-subgroup of E and so $\tilde{P}\supseteq O_p(\tilde{H})$ in this case. Suppose then that $K\not\subseteq O(H)$. Since $\bar{L}$ is the unique minimal normal subgroup of $\bar{H}$, K must cover $\bar{L}$. Since $\bar{H}=\bar{L}O(C_{\bar{H}}(\bar{S}))$, we see in this case that $F=O(H)O(C_H(S))$ covers $O_p(\tilde{H})$. However, by Lemmas 5.1 and 5.4, $C_P(Z)$ is a Sylow p-subgroup of $O(C_H(Z))$. Since $C_H(Z)$ has a normal 2-complement, $O(C_H(Z))\cap C_H(S)=O(C_H(S))$ and consequently $C_P(S)$ is a Sylow p-subgroup of $O(C_H(S))$. Since $P\cap O(H)$ is a Sylow p-subgroup of $O(H)$, it follows therefore that $P\cap F$ is a Sylow p-subgroup of F. Thus $\tilde{P}$ contains $O_p(\tilde{H})$ in this case as well. We conclude that condition (b) in the definition of p-stability with respect to P holds when $\bar{H}$ contains a simple normal subgroup of odd index.

On the other hand, if $\bar{H}\cong GL(3, 2)\cdot E_8^{(2)}$, it is immediate that $P\subseteq O(H)$ and that $O(H)$ covers $O(\tilde{H})$ in both cases $K\subseteq O(H)$ and $K\not\subseteq O(H)$. Since P is a Sylow p-subgroup of $O(H)$, the desired conclusion $\tilde{P}\supseteq O_p(\tilde{H})$ follows at once.

Thus we may assume that Proposition 5.3(i) holds. In view of assumption (a), we have that $\bar{H}\cong M/O(M)$, so we can identify $\bar{H}$ with $\bar{M}$. Again let K be normal in H and set $\tilde{H}=H/K$. If $K\subseteq O(H)$ or $O(H)\langle z\rangle$, it follows exactly as in the corresponding part of the first case that $\tilde{P}\supseteq O_p(\tilde{H})$. Since $\bar{H}=\bar{M}=\bar{L}_1\bar{L}_2O(C_{\bar{H}}(\bar{S}))\bar{S}$ by Lemma 4.7(i), it follows likewise as in the corresponding part of the first case, if K covers $\bar{L}_1\bar{L}_2$, that $\tilde{P}\supseteq O_p(\tilde{H})$. Hence we may assume that $K\not\subseteq O(H)\langle z\rangle$ and that K does not cover $\bar{L}_1\bar{L}_2$. But any normal subgroup of $\bar{H}$ which contains $\langle\bar{z}\rangle$ properly necessarily contains either $\bar{L}_1'$ or $\bar{L}_2'$. Note that $\bar{L}_i=\bar{L}_i'$ if $q_i>3$, $i=1$ or 2.

If $q_i>3$ for both $i=1$ and 2, it follows from the structure of $\bar{H}$ that $O(H)O(C_H(S))$ covers $O_p(\tilde{H})$ and condition (b) follows once again as in the corresponding part of the first case. Likewise if $p\neq 3$, $O(H)O(C_H(S))$ covers $O_p(\tilde{H})$ and again condition (b) holds. Hence we may assume that $p=3$ and that, say, $q_1=3$. If also $q_2=3$, then clearly $P\subseteq O(H)$ and so P is a Sylow p-subgroup of $O(H)$. But then P is normal in a Sylow p-subgroup of H and so the first alternative of condition (b) in the definition of p-stability with respect to P holds; so we can also suppose that $q_2>3$. Hence if K covers $\bar{L}_1$, it again follows that $O(H)O(C_H(S))$ covers $O_p(\tilde{H})$, whence condition (b) holds as before. Likewise we reach the same conclusion if K covers $\bar{L}_2$, but does not cover $\bar{L}_1'$.

Thus it remains to consider the case that $p=3$, $q_1=3$, $q_2>3$, and K covers $\bar{L}_1'$, but not $\bar{L}_1$. In this case we set $\bar{H}_0=\bar{L}_1'\bar{L}_2O(C_{\bar{H}}(\bar{S}))\bar{S}$. Then $|\bar{H}:\bar{H}_0|=3$ and if H_0 denotes the inverse image of $\bar{H}_0$ in H, we see that $|H:H_0|=3$, $O(H)PS\subseteq H_0$, H_0 is p-constrained, and $O_{p'}(H_0)$

$=O_{p'}(H)$. Moreover, if K_0 is any normal subgroup of H_0, it follows as above that PK_0/K_0 contains $O_p(H_0/K_0)$, so H_0 satisfies condition (b) and likewise it satisfies condition (a) in the definition of p-stability with respect to P. Finally we note that both $\bar{L}_2$ and $O(C_{\bar{H}}(\bar{S}))$ centralize $\bar{L}_1$. Since clearly $\bar{P}\subseteq\bar{L}_2O(C_{\bar{H}}(\bar{S}))$, we have that $\bar{H}=\bar{H}_0N_{\bar{H}}(\bar{P})$. Hence by the Frattini argument, $H=H_0N_H(P)$ and so H_0 satisfies the various side conditions stated for H_0 in the exceptional case.

Hence to complete the proof of the lemma, we have now to show that if P_0 is any nontrivial subgroup of P such that $O_{p'}(H)P_0\lhd H$ and if A is any p-subgroup of P such that $[P_0, A, A]=1$, then

$$(2) \qquad AC_H(P_0)/C_H(P_0) \subseteq O_p(N_H(P_0)/C_H(P_0))$$

with a corresponding statement with H_0 in place of H in the exceptional case. However, if we verify this assertion for H itself, it is clear that it will also hold for H_0 and so it is enough to treat the case of H itself.

We next reduce to the case that H has the form given in Proposition 5.3(i). Suppose then that $\bar{H}$ satisfies Proposition 5.3(iv). Lemma 5.4 implies that $\bar{P}\subseteq O(C_{\bar{H}}(\bar{Z}))\subseteq\bar{H}_1=C_{\bar{H}}(\bar{z})$. Setting H_1 equal to the inverse image of $\bar{H}_1$ in H, we have that $A\subseteq PS\subseteq H_1$. Furthermore, our minimal choice of G implies that $O(\bar{H}_1)=1$ and consequently $O(H_1)=O(H)$ and $O_{p'}(H_1)=O_{p'}(H)$. Clearly the p-constraint of H also implies the p-constraint of H_1. Since we also have that $H_1/O(H_1)\cong C_{\bar{H}}(\bar{z})\cong M/O(M)$, we see that H_1 satisfies all the conditions which H would satisfy when H has the form given in Proposition 5.3(i).

We argue now that if (2) holds for H_1, then it holds for H. Suppose then that (2) is false for H. By a standard argument, a suitable homomorphic image $\tilde{H}$ of H with $O_p(\tilde{H})=1$ is faithfully represented on a vector space W over $GF(p)$ in such a way that $\tilde{A}\neq 1$ and $[W, \tilde{A}, \tilde{A}]=1$. Since $\bar{H}=H/O(H)$ is either isomorphic to $GL(3, 2)\cdot E_8^{(2)}$ or has a normal simple subgroup of odd index, it follows that the kernel of this representation is contained in $O(H)$. Since $O(H)=O(H_1)$ and $O(C_{\bar{H}_1}(\bar{z}))=1$, this implies that $O_p(\tilde{H}_1)=1$. However, since we are assuming that (2) holds for H_1, we have that $\tilde{A}\subseteq O_p(\tilde{H}_1)$, so $\tilde{A}=1$, which is a contradiction. Thus it will suffice to treat the case that $\bar{H}=H/O(H)\cong M/O(M)$.

As in the proof of [**30**, Lemma 11.10], the question of the validity of (2) can be reduced, again by a standard argument, to the following situation: A vector space U over $GF(p)$ acted on faithfully and irreducibly by a group J containing A with $O(J)$ a p'-group in the center of J and with $J/O(J)$ isomorphic to a homomorphic image of the normal closure of $\bar{A}$ in $\bar{H}=H/O(H)$. To establish the desired conclusion, we must show under these conditions that $[U, A, A]\neq 1$.

We assume the contrary, in which case J involves $SL(2, p)$. We claim that $J/O(J)$ is actually isomorphic to a *subgroup* of $\bar{H}$. Indeed, as $\bar{H}\cong M/O(M)$, either this is the case or $J/O(J)$ is isomorphic to a homomorphic image of $\bar{H}/\langle\bar{z}\rangle$. However, in the latter case, it would follow from the structure of $M/O(M)$ that J had a normal subgroup of odd index with Sylow 2-subgroups that were dihedral or the direct product of two dihedral groups. But then we see that J cannot involve $SL(2, p)$. This proves our assertion. Thus we can identify $\bar{J}=J/O(J)$ with the normal closure of $\bar{A}$ in $\bar{H}$. For simplicity, we also identify $\bar{H}$ with $\bar{M}$.

Since $\bar{J}$ is the normal closure of $\bar{A}$ in $\bar{H}$ and since $\bar{H}/\bar{L}_1\bar{L}_2=\bar{M}/\bar{L}_1\bar{L}_2$ is abelian, we see that $\bar{J}=\overline{LA}$, where $\bar{L}=\bar{L}_1$, $\bar{L}_2$, or $\bar{L}_1\bar{L}_2$. We argue first that $\bar{A}\subseteq\bar{L}$. Indeed, suppose not. Then as $O(J)$ is a p'-group, $\bar{A}$ does not centralize $\bar{L}$ and so does not centralize one of the components of $\bar{L}$, which, for definiteness, we can take to be $\bar{L}_1$. Now $\bar{P}$ is a maximal $\bar{S}$-invariant p-subgroup of $\bar{H}$ and so $\bar{P}\cap\bar{L}_1$ is a maximal $\bar{Q}_1$-invariant p-subgroup of $\bar{L}_1$. Thus $\bar{P}=(\bar{P}\cap\bar{L}_1)C_{\bar{P}}(\bar{Q}_1)$. If $\bar{P}\cap\bar{L}_1\neq 1$, it follows from standard properties of the group $\Gamma L(2, q_1)$ that $C_{\bar{L}_1}(\bar{P}\cap\bar{L}_1)$ is a $\bar{P}$-invariant cyclic group and that any element of $\bar{P}-C_{\bar{P}}(\bar{L}_1)(\bar{P}\cap\bar{L}_1)$ normalizes, but does not centralize a Sylow r-subgroup of $C_{\bar{L}_1}(\bar{P}\cap\bar{L}_1)$ for some odd prime $r\neq p$. On the other hand, if $\bar{P}\cap\bar{L}_1=1$ and $\bar{Q}_0$ is a normal subgroup of $\bar{Q}_1$ of order 4, we conclude similarly that every element of $\bar{P}-C_{\bar{P}}(\bar{L}_1)$ normalizes, but does not centralize a Sylow r-subgroup of $C_{\bar{L}_1}(\bar{Q}_0)$ for some odd prime $r\neq p$. Since $O(J)$ is a p'-group and $\bar{A}\subseteq\bar{P}$ does not centralize $\bar{L}_1$, it follows that some element a of $A^{\#}$ normalizes, but does not centralize an r-subgroup of J for some odd prime $r\neq p$. The Hall-Higman theorem now yields that $[U, a, a]\neq 1$, contrary to our present assumption $[U, A, A]=1$. Thus $\bar{A}\subseteq\bar{L}$, as asserted.

Since $\bar{L}$ is the normal closure of $\bar{A}$ in $\bar{H}$ and $\bar{A}\subseteq\bar{P}$, clearly $\bar{P}\cap\bar{L}_i\neq 1$ if $\bar{L}_i\subseteq\bar{L}$, $i=1$ or 2. But now as $\bar{P}\cap\bar{L}_i$ is a maximal $\bar{Q}_i$-invariant subgroup of $\bar{L}_i$, it follows from the structure of $SL(2, q_i)$ that $q_i\neq p^{m_i}$ and also if $p=3$, that $q_i\neq 5$. If L_i denotes the inverse image of $\bar{L}_i$ in J, the argument at the end of [**55**, Proposition 2.6.1] shows that no p-element of L_i has a quadratic minimal polynomial on U. Hence we reach a contradiction unless $\bar{L}=\bar{L}_1\bar{L}_2$. However, in this case, L_i centralizes $P\cap L_j$ for $i\neq j$ and $\tilde{L}_i=L_i(P\cap L_j)/P\cap L_j$ is faithfully represented on $U_j=C_U(P\cap L_j)$. Again the argument of [**55**] implies that this representation is p-stable. Since $\tilde{A}\neq 1$, we conclude that $[U_j, \tilde{A}, \tilde{A}]\neq 1$, contrary to the fact that $[U, A, A]=1$. Thus (2) holds in all cases and the lemma is finally proved.

6. **Covering p-local subgroups.** By assumption, $O(M)\neq 1$. We let π be the set of primes dividing $|O(M)|$. For each p in π we shall now construct a p-local subgroup K_p of G with the following properties:

(a) K_p contains both S and an S-invariant Sylow p-subgroup of $O(M)$;

(b) K_p covers $M/O(M)$;

(c) $K_p/O(K_p)$ is fusion-simple.

We call K_p a *covering p-local* subgroup of G.

We fix a prime p in π, we let P_1 be a maximal S-invariant p-subgroup of M containing a maximal S-invariant p-subgroup of $C=C_G(Z)$ and we set $P_0=P_1\cap O(M)$, so that P_0 is a Sylow p-subgroup of $O(M)$ and $P_0\neq 1$. We fix this notation and also recall that $Z=\langle z_1, z_2\rangle$, $N=N_G(Z)$, and $C=C_G(Z)$.

We first prove

LEMMA 6.1. *There exists a maximal S-invariant p-subgroup Q of G such that $N_G(Q)$ covers $N/O(N)$ and $Q\cap P_1\neq 1$.*

PROOF. We claim that $C_{P_1}(z_1)=C_{P_1}(Z)\neq 1$. This is clearly the case if z_1 centralizes P_0 as $P_0\neq 1$, so assume z_1 does not centralize P_0. By the Frattini argument, $N_M(P_0)$ covers $M/O(M)$. But now observing the structure of M, we see that the image of z_1 in $N_M(P_0)/C_M(P_0)$ is not in the center of this group. It follows that z_1 does not invert P_0 and hence that $C_{P_1}(z_1)\neq 1$, so our assertion holds in this case as well.

By Lemma 5.1(ii), C has a normal 2-complement. It follows therefore from our choice of P_1 that $Q_1=C_{P_1}(Z)$ is an S-invariant Sylow p-subgroup of $O(C)=O(N)$. Since $Q_1\neq 1$, $N_G(Q_1)$ is thus a p-local subgroup of G which covers $N/O(N)$.

Let H be a p-local subgroup of G such that

(a) H covers $N/O(N)$ and contains S;

(b) $O_p(H)\supseteq Q_1$;

(c) Subject to (a) and (b), $|O_p(H)|$ is maximal;

(d) Subject to (a), (b), and (c), a maximal S-invariant p-subgroup of H has maximal order.

Clearly such an H exists. We shall argue that $O_p(H)$ is, in fact, a maximal S-invariant p-subgroup of G.

Let Q be a maximal S-invariant p-subgroup of H and set $Q_0=Q\cap O(H)$. Then Q_0 is a Sylow p-subgroup of $O(H)$. By the Frattini argument, $N_H(Q_0)$ covers $H/O(H)$ and so covers $N/O(N)$. Hence $H_0=N_G(Q_0)$ covers $N/O(N)$, contains S, and $Q_0\subseteq O_p(H_0)$. Since $Q_0\supseteq O_p(H)\supseteq Q_1$, we conclude from our maximal choice of H that $Q_0=O_p(H)$.

Now set $\bar{H}=H/O(H)$. By Lemma 5.4, $\bar{Q}$ is a Sylow p-subgroup of $O(C_{\bar{H}}(\bar{Z}))$. However, Q_1 is a Sylow p-subgroup of $O(C)$ and hence of $O(C_H(Z))$. Since $Q_1\subseteq O(H)$ and $O(C_H(Z))$ maps onto $O(C_{\bar{H}}(\bar{Z}))$, it follows that $\bar{Q}_1=1$. Thus $Q=Q_0=O_p(H)$. But now we see from our maximal choice of H that Q must be a maximal S-invariant p-subgroup of $N_G(Q)$, which implies that Q is, in fact, a maximal S-invariant p-subgroup of G.

LEMMA 6.2. *One of the following holds*:
(i) *P_1 is a maximal S-invariant p-subgroup of G*; *or*
(ii) *$N_G(P_0)$ is p-constrained and $O_{p'}(N_G(P_0))\subseteq O(N_G(P_0))$.*

PROOF. We set $H=N_G(P_0)$, so that H covers $M/O(M)$ and $H\supseteq P_1S$. Let P_2 be a maximal S-invariant p-subgroup of H containing P_1 and P_3 a maximal S-invariant p-subgroup of G containing P_2. By a standard use of Thompson's $A\times B$-lemma [**60**, Theorem 5.3.4] (cf. [**31**, Lemma 8.7]), z centralizes P_2 if and only if it centralizes P_3. Since $z\in Z$, it follows from Lemma 5.4 that $C_{P_2}(z)$ covers

$$P_2/P_2\cap O(H).$$

Hence z centralizes P_2 if and only if it centralizes $P_2\cap O(H)$.

If z centralizes P_3, then $P_3=P_1$ as $P_3\supseteq P_1$ and P_1 is a maximal S-invariant p-subgroup of $M=C_G(z)$. Thus (i) holds in this case. In the contrary case, z does not centralize $P_2\cap O(H)$ by the preceding paragraph and so by a standard argument, z does not centralize $Q_0=P_2\cap O_{p',p}(O(H))$. Since z leaves Q_0 invariant, this in turn implies that $z\notin O_{p'}(H)$. Since $\langle z\rangle=Z(S)$ and S is a Sylow 2-subgroup of H, it follows that $|O_{p'}(H)|$ is odd and hence that $O_{p'}(H)\subseteq O(H)$. For the same reason, $C_H(Q_0)\subseteq O(H)$ and hence $C_H(Q_0)\subseteq O_{p',p}(O(H))$ $=O_{p',p}(H)$. Thus H is p-constrained and (ii) holds.

We recall that $R=Q_1Q_2\cong Q_{2^{n+1}} * Q_{2^{n+1}}$ with $Q_1=\langle ab^{-1}, z_1t\rangle$ and $Q_2=\langle ab, z_1tu\rangle$ and, moreover, that R is a Sylow 2-subgroup of M'. We next prove

LEMMA 6.3. *If R centralizes P_0, then $N_G(P_0)$ is a covering p-local subgroup of G.*

PROOF. We set $A=\langle u, Z\rangle$, so that A is an elementary abelian subgroup of R of order 8. By Lemma 5.2, $C_G(A)=A\times O(C_G(A))$ and $N_G(A)/C_G(A)\cong GL(3, 2)$. Moreover, by Lemma 4.7(ii), $O(C_G(A))$ $\subseteq O(M)C_G(S)$. Since $A\subseteq R$ centralizes P_0 by hypothesis and P_0 is a Sylow p-subgroup of $O(M)$, it follows that an S-invariant Sylow p-subgroup P of $O(M)C_G(S)$ containing p_0 is a Sylow p-subgroup of $O(C_G(A))$. Furthermore, $P_0\lhd P$ and S centralizes P/P_0, so R centralizes P.

By the Frattini argument $N_G(P)$ covers $\bar{H}=N_G(A)/C_G(A) \cong GL(3, 2)$. Moreover, $R_1=N_R(A)\supset A$ and so $\bar{R}_1\neq 1$. Since $\bar{H}$ is simple and $\bar{R}_1$ centralizes P, it follows that $C_G(P)$ also covers $\bar{H}$. Since $P_0\subseteq P$, also $C_G(P_0)$ covers $\bar{H}$. We conclude therefore that $K=N_G(P_0)$ covers both $N_G(A)/C_G(A)$ and $M/O(M)$. But now K must satisfy Proposition 5.3(iv) and so $K/O(K)$ is fusion-simple. Thus K is a covering p-local subgroup and the lemma is proved.

We now prove the existence of covering p-local subgroups in each of the two cases of Lemma 6.2.

LEMMA 6.4. *If P_1 is a maximal S-invariant p-subgroup of G, then $N_G(P_0)$ is a covering p-local subgroup of G.*

PROOF. Let Q be a maximal S-invariant p-subgroup of G which satisfies the conditions of Lemma 6.1. By Proposition 5.5, P_1 is conjugate to Q by an element of $C_G(S)$. Since z centralizes P_1, z thus centralizes Q. Since $N_G(Q)$ covers $N/O(N)$ and N contains a 3-element which cyclically permutes the involutions of Z, it follows that Z centralizes Q. Thus Z centralizes P_1. In particular, Z centralizes P_0. But $N_M(P_0)$ covers $M/O(M)$ and it is immediate from the structure of M that the normal closure of Z in $N_M(P_0)$ contains R. Thus R centralizes P_0 and now the desired conclusion follows from the preceding lemma.

LEMMA 6.5. *If $N_G(P_0)$ is p-constrained and $O_{p'}(N_G(P_0))\subseteq O(N_G(P_0))$, then $N_G(Z(J(P)))$ is a covering p-local subgroup of G for some maximal S-invariant p-subgroup P of G.*

PROOF. We set $H=N_G(P_0)$, so that H covers $M/O(M)$, H is p-constrained, $H\supseteq P_1$, and $O_{p'}(H)\subseteq O(H)$. We also let P^* be an SP_1-invariant Sylow p-subgroup of $O_{p',p}(O(H))=O_{p',p}(H)$. Our conditions imply that z does not centralize P^*. Moreover, by the Frattini argument, $N_G(P^*)$ is a p-local subgroup of G which covers $H/O(H)$ and hence covers $M/O(M)$. In addition, $N_G(P^*)$ contains SP_1.

We now let K be a p-local subgroup of G such that

(a) K covers $M/O(M)$ and contains S;

(b) $O_p(K)\supseteq P^*$ and $K\supseteq P_1$;

(c) Subject to (a) and (b), $|O_p(K)|$ is maximal;

(d) Subject to (a), (b), (c), a maximal S-invariant p-subgroup of K has maximal order.

Since $N_G(P^*)$ satisfies conditions (a) and (b), such a p-local subgroup K exists. Moreover, as in Lemma 6.1, it follows by the Frattini argument that $O_p(K)$ is a Sylow p-subgroup of $O(K)$. Since $P^*\subseteq O_p(K)$ and z does not centralize P^*, z does not centralize $O_p(K)$. Since

$\langle z\rangle = Z(S)$, we conclude at once from this that $C_K(O_p(K)) \subseteq O(K)$ and that $O_{p'}(K) \subseteq O(K)$. Together these conditions imply that K is p-constrained. We see then that K satisfies all the hypotheses of Lemma 5.6. Hence if P is a maximal S-invariant p-subgroup of K containing P_1, we conclude that

$$K = O_{p'}(K)N_K(Z(J(P))).$$

We set $J = N_G(Z(J(P)))$ and now investigate J. Since K covers $M/O(M)$ and $O_{p'}(K) \subseteq O(K)$, it is immediate that $K \cap J$ and hence also J covers $M/O(M)$. Since P is S-invariant, also $PS \subseteq J$. But $O_p(K) \lhd P$ and $P^* \subseteq O_p(K)$, so, in particular, $P^* \subseteq O_p(K) \subseteq J$.

We claim next that $Q_0 = O_p(K) \subseteq O(J)$. Indeed, set $\bar{J} = J/O(J)$. Since J covers $M/O(M)$, J satisfies the conditions of either Proposition 5.3(i) or (iv). Setting $K_0 = K \cap J$, it follows in the first case that $\bar{K}_0 = \bar{J}$ as K_0 covers $M/O(M)$. But $Q_0 = O_p(K) \lhd K_0$ and so $\bar{Q}_0 \lhd \bar{K}_0 = \bar{J}$. Since $O(\bar{J}) = 1$, this forces $\bar{Q}_0 = 1$ and so $Q_0 \subseteq O(J)$ in this case. In the second case, $\bar{J}$ is fusion-simple and $\bar{J}$ satisfies the conclusion of Theorem B by the minimality of G, so $O(C_{\bar{J}}(\bar{z})) = 1$ and $C_{\bar{J}}(\bar{z}) \cong M/O(M)$. Since K_0 covers $M/O(M)$, it follows that $\bar{K}_0 = C_{\bar{J}}(\bar{z})$ and that $O(\bar{K}_0) = 1$. But $\bar{Q}_0 \lhd \bar{K}_0$ as $Q_0 \lhd K_0$, so $\bar{Q}_0 = 1$ and $Q_0 \subseteq O(J)$ in this case as well.

Finally, let $\tilde{P}$ be a maximal S-invariant p-subgroup of J containing P and consider $L = N_G(\tilde{P} \cap O(J))$. By the Frattini argument, L covers $J/O(J)$ and so covers $M/O(M)$. Furthermore, L contains $\tilde{P}S$ and $O_p(L) \supseteq \tilde{P} \cap O(J)$. But $\tilde{P} \cap O(J) \supseteq P \cap O(J) \supseteq O_p(K)$ by the preceding paragraph and so $O_p(L) \supseteq O_p(K) \supseteq P^*$. Since $\tilde{P} \supseteq P \supseteq P_1$, we see then that L is a p-local subgroup of G which satisfies conditions (a) and (b) above and which is at least as large as K in our ordering. But now our maximal choice of K forces $O_p(L) = O_p(K)$ and $\tilde{P} = P$. Since $Z(J(P))$ is characteristic in P, we conclude at once from the definition of J and $\tilde{P}$ that P is, in fact, a maximal S-invariant p-subgroup of G.

Now we can easily show that $J = N_G(Z(J(P)))$ is a covering p-local subgroup of G. Let Q be a maximal S-invariant p-subgroup of G satisfying the conditions of Lemma 6.1, so that $Q \cap P_1 \neq 1$ and $N_G(Q)$ covers $N/O(N)$. Since $P \supseteq P_1$, $Q \cap P \neq 1$ and so by Proposition 5.5, Q is conjugate to P by an element of $C_G(S)$. But then $N_G(P)$ also covers $N/O(N)$ and consequently so does J. Since J covers $M/O(M)$ and contains S, we conclude therefore from Proposition 5.3 that $J/O(J)$ is fusion-simple. Since $J \supseteq P_0$, J is thus a covering p-local subgroup of G, as asserted.

Combining Lemmas 6.2, 6.4, and 6.5, we obtain the objective of this section.

PROPOSITION 6.6. *G possesses a covering p-local subgroup for each prime p in π.*

7. **Proof of Theorem B.** Since G is a minimal counterexample to Theorem B, we have that $O(M) \neq 1$. We shall now contradict this conclusion by showing that, in fact, $O(M) = 1$. We know from Lemmas 4.1(iii) and 4.8 that G is a connected group of 2-rank 3 in which the centralizer of every involution is 2-generated. Since G is fusion-simple we also have that $O(G) = 1$. Hence if we can demonstrate that G is balanced, Theorem 2.1 will be applicable and will yield that $O(C_G(z)) = O(M) = 1$, giving the desired contradiction.

Thus Theorem B will be proved, once we establish the following result:

PROPOSITION 7.1. *G is balanced.*

PROOF. We must show that $O(C_G(x)) \cap C_G(y) \subseteq O(C_G(y))$ for every pair of commuting involutions x, y of G. Since G has only one conjugacy class of involutions, we may assume that $x = z$ and that $y \in S$. We set $D = C_{O(M)}(y)$ and we must prove that $D \subseteq O(C_G(y))$. It will clearly suffice to show that for each prime p dividing $|D|$, a Sylow p-subgroup of D is contained in $O(C_G(y))$.

By Proposition 5.6, G possesses a covering p-local subgroup $K = K_p$. In particular, K contains S as well as an S-invariant Sylow p-subgroup P_0 of $O(M)$. Clearly $D_0 = C_{P_0}(y)$ is then a Sylow p-subgroup of D, so we need only show that $D_0 \subseteq O(C_G(y))$.

Setting $\overline{K} = K/O(K)$, we have that $\overline{K}$ is fusion-simple. Since $C_{\overline{K}}(\bar{z})$ covers $M/O(M)$ and since the centralizer of an involution in M_{12} or $GL(3, 2) \cdot E_8^{(2)}$ does not involve $SL(2, q_1) * SL(2, q_2)$, $\overline{K}$ is not isomorphic to one of these groups. Hence by our minimal choice of G, $\overline{K}$ has one class of involutions and $O(C_{\overline{K}}(\bar{z})) = 1$. But clearly $P_0 \subseteq O(M) \cap K = O(C_G(z)) \cap K \subseteq O(C_K(z))$. Since $O(C_K(z))$ maps onto $O(C_{\overline{K}}(\bar{z})) = 1$, we conclude that $P_0 \subseteq O(K)$.

On the other hand, $C_{\overline{K}}(\bar{y}) \cong C_{\overline{K}}(\bar{z}) \cong M/O(M)$ as $\overline{K}$ has only one class of involutions. Hence if we set $H = C_G(y)$, we see that $C_K(y)$ covers $H/O(H)$ and hence that $H = O(H)(K \cap H)$. This implies that $O(K \cap H) \subseteq O(H)$. But clearly $O(K) \cap H \subseteq O(K \cap H)$ and so $O(K) \cap H \subseteq O(H)$. However, as $P_0 \subseteq O(K)$, $D_0 = C_{P_0}(y) = P_0 \cap H \subseteq O(K) \cap H$, whence $D_0 \subseteq O(H) = O(C_G(y))$, as required. This completes the proof of Theorem B.

APPENDIX

8. **Fusion of involutions.** In this section, we establish the three un-

published results of Fong, stated without proof in §4; namely, Theorem 4.3 and Lemmas 4.4 and 4.5.

We let S be a 2-group of type $G_2(q)$, q odd. We preserve the previous notation. We shall now list some easily verified additional properties of S. However, to do so, we require a preliminary definition. Let $D=D_1\times D_2$ be the direct product of two dihedral 2-groups D_1, D_2 and let Y be a 2-group of the form $Y=D\langle y\rangle$, where $|y|=2$, $D_i \lhd Y$, and $F_i=D_i\langle y\rangle$ is dihedral, $i=1, 2$. Under these conditions we say that Y is the *crown product* of F_1 and F_2 and write $Y=F_1\wedge F_2$. We note that in the special case that $|F_i|=8$, $i=1, 2$, $Y\cong(Z_2\times Z_2)\int Z_2$.

LEMMA 8.1. *The following conditions hold*:

(i) *S has seven conjugacy classes of involutions, represented by z, z_1, t, tu, u, uab, ua*:

(ii)
$$\begin{aligned} C_S(z_1) &= T = \langle W, u\rangle; \\ C_S(t) &= \langle t\rangle \times \langle u, ab\rangle \cong Z_2 \times D_{2^{n+1}}; \\ C_S(tu) &= \langle tu\rangle \times \langle u, ab^{-1}\rangle \cong Z_2 \times D_{2^{n+1}}; \\ C_S(u) &= \langle u\rangle \times \langle t, Z\rangle \cong Z_2 \times D_8; \\ C_S(uab) &= \langle uab\rangle \times \langle t, Z\rangle \cong Z_2 \times D_8; \\ C_S(ua) &= \langle ua, Z\rangle \cong E_8. \end{aligned}$$

(iii) *If $\bar{S}=S/\langle z\rangle$, then $\bar{R}=\bar{Q}_1\times\bar{Q}_2\cong D_{2^n}\times D_{2^n}$ and $\langle\bar{Q}_1, \bar{u}\bar{b}\rangle\cong\langle\bar{Q}_2, \bar{u}\bar{b}\rangle\cong D_{2^{n+1}}$.*

(iv) *T is the unique maximal subgroup of S with a noncyclic center.*

(v) *All involutions of $\langle t, W\rangle - W$ and $\langle tu, W\rangle - W$ are conjugate to t and tu respectively.*

(vi) *$\langle u, a^2, b^2\rangle$ is the normal closure of u in S and $S/\langle u, a^2, b^2\rangle\cong D_8$.*

(vii) *$S/\langle z\rangle\cong D_{2^{n+1}}\wedge D_{2^{n+1}}$.*

Now let G be a finite group with Sylow 2-subgroup S. Our first result will establish Lemma 4.4.

LEMMA 8.2. *If G has no isolated involution, then $N_G(Z)/C_G(Z)\cong S_3$.*

PROOF. We first show that $z\sim z_1$. Suppose false, in which case Glauberman's Z^*-theorem [47] together with Lemma 8.1(i) implies that z is conjugate to y, where $y=t$, tu, u, uab, or ua. If $y\neq ua$, then Lemma 8.1(ii) implies that $z\in\Omega_1(C_S(y)')$. Hence if we choose $g\in G$ such that

$$y^g = z \quad\text{and}\quad C_S(y)^g \subseteq S,$$

then $z^g = z_1$ or z_2, by Lemma 4.1(ii). In either case $z \sim z_1$, contrary to our present assumption. Hence $y = ua$.

Choose h in G such that

$$(ua)^h = z \quad \text{and} \quad C_S(ua)^h = \langle ua, Z\rangle^h \subsetneq S.$$

Since $uaz_1 \sim ua$ in S, $(uaz_1)^h$ is not conjugate in S to t, tu, u, or uab or else z is conjugate to one of these elements, contrary to what we have just shown. Clearly $(uaz_1)^h \neq z$ and so there exists s in S such that $(uaz_1)^{hs} = ua$. But then

$$z_1^{hs} = (ua \cdot uaz_1)^{hs} = z \cdot ua \sim ua \sim z,$$

which is again a contradiction. Thus $z \sim z_1$, as asserted.

Now choose $k \in G$ such that

$$z_1^k = z \quad \text{and} \quad C_S(z_1)^k = T^k \subseteq S.$$

But T is the unique maximal subgroup of S with a noncyclic center by Lemma 8.1(iv) and so $T^k = T$. Hence $Z^k = Z$ as $Z = Z(T)$. From this, our lemma follows at once.

Henceforth we assume that G is fusion-simple and we again set $M = C_G(z)$. We next prove

LEMMA 8.3. *G has exactly one or two conjugacy classes of involutions. In the latter case, z, t, are not conjugate and t, tu are conjugate.*

PROOF. By Lemma 8.1(v), all involutions of $\langle t, W\rangle - W$ are conjugate to t in S. Since $Z = \Omega_1(W)$, it follows from the preceding lemma that all involutions of W are conjugate to z in G. Since $\langle t, W\rangle$ is maximal in S and G has no normal subgroups of index 2, Thompson's fusion lemma [**45**, Lemma 5.38] implies that every involution of S is conjugate in G to t or z. Similarly considering $\langle tu, W\rangle$, every involution of S is conjugate to tu or z. Together these two assertions imply the lemma.

In view of Lemma 8.1(vii), we shall also need a general property of groups with a Sylow 2-subgroup $Y \cong D_{2^{n+1}} \wedge D_{2^{n+1}}$. Since the same result is also needed in another paper [**34**], we shall make the proof independent of the present context. We know that Y is of the form $D\langle y\rangle$, where

$$D = \langle a_1, b_1\rangle \times \langle a_2, b_2\rangle \cong D_{2^n} \times D_{2^n}$$

with $a_i^{2^{n-1}} = b_i^2 = y^2 = 1$ and $a_i^y = a_i^{-1}$, $b_i^y = a_i b_i$, $i = 1, 2$. Since $Y \cong (Z_2 \times Z_2) \int Z_2$ if $n = 2$, the result we require has already been established in this case [30, Lemma 4.4]. Thus we shall limit ourselves here to the case $n \geqq 3$. We let e_i be the central involution of $\langle a_i, b_i \rangle$, $i=1, 2$, and set $E=\langle e_1, e_2 \rangle$, so that $E=Z(Y)$. We fix this notation for the next two lemmas.

To establish our desired assertion, we first derive a general property of groups with Sylow 2-subgroup D; that is, the direct product of two nonabelian dihedral groups. Although the specific result we need was not proved in [33], some of the arguments of [33] will easily yield it.

LEMMA 8.4. *If X is a group with Sylow 2-subgroup D in which $E \subseteq Z(X)$, then X has a normal 2-complement.*

PROOF. Setting $K=O^2(X)$, we shall argue first that $K \cap D$ contains only central involutions of D. Representatives of the conjugacy classes of noncentral involutions of D are b_1, a_1b_1, b_2, a_2b_2, b_1b_2, $a_1b_1b_2$, $b_1a_2b_2$, and $a_1b_1a_2b_2$.

Suppose first that $b_1 \in K$. By Thompson's fusion lemma, b_1 must then be conjugate to an involution u of $\langle a_1 \rangle \times \langle a_2, b_2 \rangle$. Since $E \subseteq Z(X)$, $u \notin E$. Hence $C_D(u)=\langle a_1, b_1 \rangle \times \langle e_2, u \rangle \cong D_{2^n} \times Z_2 \times Z_2$. We see then that $|C_D(b_1)| = |C_D(u)|$ and that b_1, u are each extremal in D. Then X contains an element x such that

$$b_1^x = u \quad \text{and} \quad C_D(b_1)^x = \langle b_1, e_1 \rangle^x \times \langle a_2, b_2 \rangle^x = C_D(u).$$

But then $(C_D(b_1)')^x = C_D(u)'$ and so $e_2^x = e_1$, which is impossible as $E \subseteq Z(X)$. The above argument clearly works for any involution d of D such that $C_D(d) \cong D_{2^n} \times Z_2 \times Z_2$ if we choose a suitable maximal subgroup of D. We conclude therefore that b_1, a_1b_1, b_2, a_2b_2 are not in K.

Suppose next that $b_1b_2 \in K$. Again by Thompson's transfer lemma, b_1b_2 is conjugate to an involution u of $\langle a_1 \rangle \times \langle a_2, b_2 \rangle$. Hence $C_D(u) = \langle a_1, b_1 \rangle \times \langle e_2, u \rangle$ and again u is extremal in D. On the other hand, $B=C_D(b_1b_2)=\langle e_1, b_1, e_2, b_2 \rangle$ is elementary of order 16 and b_1b_2 is not extremal in D. In this case there exists x in X such that

$$(b_1b_2)^x = u \quad \text{and} \quad B^x \subseteq C_D(u).$$

By [33, Lemma 3.1], two elementary abelian subgroups of D of order 16 are conjugate in X if and only if they are conjugate in D. (We note that the proof of the lemma does not require X to have no normal subgroups of index 2.) Since $B^x \subseteq C_D(u)$, it follows that for some element d in D, $B^{xd}=B$. Thus b_1b_2 and u^d are conjugate in $N=N_X(B)$.

By the structure of D, N has Sylow 2-subgroups of the form $D_8 \times D_8$. Since $\mathrm{Aut}(D_8 \times D_8)$ is a 2-group, it follows at once, using the Frattini argument, that a Sylow 2-subgroup of $\overline{N} = N/C_X(B)$ is a four group and is in the center of its normalizer in $\overline{N}$. Thus $\overline{N}$ has a normal 2-complement $\overline{Q}$. Suppose $\overline{Q} \neq 1$. Since $E \subseteq Z(X)$, the only possibility is that $|\overline{Q}| = 3$, $[\overline{Q}, E] = 1$, and $[\overline{Q}, B]$ is a four group disjoint from E. But $[\overline{Q}, B] \lhd N$ as $\overline{Q} \lhd \overline{N}$ and so $[\overline{Q}, B] \cap E \neq 1$, a contradiction. Thus $\overline{Q} = 1$ and N has a normal 2-complement. We conclude that $N = N_X(B) = N_D(B)C_X(B)$.

It follows now that u^d is conjugate to b_1b_2 in $N_D(B)$ and so $u^d = b_1b_2$, $b_1e_1b_2$, $b_1b_2e_2$, or $b_1e_1b_2e_2$. However, $u^d \in \langle a_1, a_2, b_2\rangle^d = \langle a_1, a_2, b_2\rangle$, which clearly contains none of the preceding four elements. This contradiction shows that $b_1b_2 \notin K$. By a similar argument we obtain that $a_1b_1b_2$, $b_1a_2b_2$, and $a_1b_1a_2b_2$ are not in K.

We have therefore finally proved that $D_0 = K \cap D$ does not contain any noncentral involution of D. We check now from the structure of D that there are two possible structures for D_0; namely, $D_0 \subseteq \langle a_1, a_2\rangle$ or $D_0 = \langle u, v\rangle$ with $\langle u, v^2\rangle = \langle u\rangle \times \langle v^2\rangle \subseteq \langle a_1, a_2\rangle$ and $v \notin \langle a_1, a_2\rangle$. (For example, $D_0 = \langle a_1, a_2b_1\rangle$ is of the second type.)

In the first case, as $E = \Omega_1(\langle a_1, a_2\rangle) \subseteq Z(X)$, Burnside's transfer theorem implies that K has a normal 2-complement. Since $K = O^2(K)$, this forces $D_0 = 1$. Thus X has a normal 2-complement in this case.

Now assume that $D_0 = \langle u, v\rangle$ is of the second form. In this case, we apply the extended form of Thompson's fusion lemma described in [**30**, §4]. Let e, f be the involutions of $\langle u\rangle$, $\langle v\rangle$ respectively, so that $e, f \in E$ and $e \neq f$. If v were conjugate in K to an element of $\langle u\rangle \times \langle v^2\rangle$, it would follow that $|u| = |v|$ and that f was conjugate to e in K, contrary to the fact that $E \subseteq Z(X)$. Since every element of $D_0 - \langle u, v^2\rangle$ has order at least $|v|$, we also see that v^{2^i} for $i \geqq 1$ is not conjugate to an element of $D_0 - \langle u, v^2\rangle$. But now we conclude from the abovementioned lemma that K has a normal subgroup of index 2, contrary to the fact that $K = O^2(K)$. This completes the proof.

Now we prove

LEMMA 8.5. *If X is a group with Sylow 2-subgroup Y, then X has a normal subgroup of index 2 with Sylow 2-subgroup D.*

PROOF. We check directly that D is the unique maximal subgroup of Y which contains no elements of order 2^n. Hence it will suffice to prove that X possesses a normal subgroup of index 2 containing no element of order 2^n. Let v be an element of Y of order 2^n, in which case $v \in Y - D$ (for example, $v = yb_1$). One checks directly from the structure of Y that $v_1 = v^{2^{n-1}} \in E \subseteq D$, that every element of $Y - D$

is either an involution or of order 2^n, and that every involution of $Y-D$ is conjugate to y in Y. But now using the extended form of Thompson's fusion lemma [**30**, §4], we see that either our assertion is valid or else v_1 is conjugate in X to an involution of $Y-D$ and hence to y from some such choice of v.

We consider the latter possibility. Since $v_1 \in E$ and $E=Z(Y)$, it follows that $C_X(y)$ contains a Sylow 2-subgroup Y_1 of X. Without loss we can assume that Y_1 contains $C_Y(y)=\langle y, E\rangle$. Suppose first that $Z(Y_1)\subseteq\langle y, E\rangle$, in which case $Z(Y_1)=\langle y, e'\rangle$, where $e'=e_1$, e_2, or e_1e_2 (as $Z(Y_1)\cong Z(Y)=E$ is a four group). But then by the structure of Y, there exists an element of order 4 in Y which normalizes, but does not centralize $Z(Y_1)$. However, this is impossible as $N_X(Z(Y_1))/C_X(Z(Y_1))$ is of odd order. Thus $Z(Y_1)\not\subseteq\langle y, E\rangle$ and as Y_1 is of 2-rank 4 we conclude that $A=\langle y, E, Z(Y_1)\rangle$ is elementary abelian of order 16.

Finally set $X_1=C_X(E)$. Then Y is a Sylow 2-subgroup of X_1 and so the preceding discussion applies to X_1 as well as to X. But clearly the involution v_1 above cannot be conjugate to y in X_1 as $v_1\in E \subseteq Z(X_1)$. Hence X_1 has a normal subgroup X_0 of index 2 with Sylow 2-subgroup D. However, as $E\subseteq Z(X_0)$, the preceding lemma implies that X_0 has a normal 2-complement and hence so does $X_1=C_X(E)$. This implies that $C=C_X(\langle E, y\rangle)=C_{O(X_1)}(y)C_Y(y)$. But $C_Y(y)=\langle y, E\rangle$ is of 2-rank 3 and consequently C has 2-rank 3. However, as $\langle y, E\rangle \subseteq A$, we have $A\subseteq C$, contrary to the fact that A has 2-rank 4.

As a first application of Lemma 8.5, we establish Lemma 4.5(ii).

LEMMA 8.6. *M has a normal subgroup of index 2 with Sylow 2-subgroup R.*

PROOF. Indeed, setting $\overline{M}=M/\langle z\rangle$, the structure of $\overline{S}$ is given in Lemma 8.1(viii). Hence if $n\geqq 3$, Lemma 8.5 implies that $\overline{M}$ has a normal subgroup of index 2 with Sylow 2-subgroup $\overline{R}$, while if $n=2$, the same conclusion holds by [**30**, Lemma 4.4]. In either case the lemma follows at once.

We can now prove Theorem 4.3.

PROPOSITION 8.7. *If $n\geqq 3$, then G has only one conjugacy class of involutions.*

PROOF. Assume false, in which case z is not conjugate to t or tu by Lemma 8.3. Hence by Lemma 8.1(ii), both t and tu are extremal in S. Thus G contains an element g such that

$$t^g = tu \quad\text{and}\quad C_S(t)^g = C_S(tu).$$

Since $\Omega_1(C_S(t)') = \Omega_1(C_S(tu)') = \langle z\rangle$, it follows that $z^g = z$, whence t and tu are conjugate in M. This implies that M does not have a normal 2-complement.

Now set $R_1 = \langle x^{-1}x^m \mid x \in S,\ x^m \in S,\ m \in M\rangle$. Then R_1 is a normal subgroup of S and is a Sylow 2-subgroup of M' by the focal subgroup theorem. But by the preceding paragraph, $u = (tu)^{-1}t \in R_1$. Since $\langle a^2, b^2, u\rangle$ is the normal closure of u in S by Lemma 8.1(vi), it follows that $\langle a^2, b^2, u\rangle \subseteq R_1 \subseteq S \cap M'$. In addition, $S \cap M' \subseteq R$ by the preceding lemma.

Suppose now that $n \geqq 3$. Consider first the case that $S \cap M' \subseteq \langle a, b, u\rangle$. We have that u inverts $\langle a, b\rangle \cap M'$ and that $S_1 = \langle a, b\rangle \cap M' \supseteq \langle a^2, b^2\rangle$. Since $n \geqq 3$, the latter condition implies that S_1 is the direct product of two cyclic groups of order at least 4. Since $S \cap M'$ is a Sylow 2-subgroup of M', we conclude now by the remark following Proposition 5.3 that M' has a normal subgroup H of index 2 with Sylow 2-subgroup S_1. But $z \in S_1$ and $z \in Z(H)$. Hence $N_H(S_1) = C_H(S_1)$ by the structure of S_1 and so H has a normal 2-complement by Burnside's transfer theorem. Thus M does as well, which is not the case.

Therefore $S \cap M' \not\subseteq \langle a, b, u\rangle$. In particular, $S \cap M' \neq \langle a^2, b^2, u\rangle$. Since $S/\langle a^2, b^2, u\rangle \cong D_8$ by Lemma 8.1(vi), it follows at once that $S \cap M' \supset \langle a^2, b^2, u, ab\rangle = \langle a^2, ab, u\rangle$. Since $S \cap M' \subseteq R$, the only possibility is that $S \cap M' = \langle a^2, ab, u, t\rangle = R$.

Setting $\overline{M} = M/\langle z\rangle$, we have that $\overline{R} \cong D_{2^n} \times D_{2^n}$, $n \geqq 3$, is a Sylow 2-subgroup of $\overline{M}'$. Since $\overline{R}$ is maximal in $\overline{S}$, $\overline{M}$ has no normal subgroup of index 4 and this implies that $O^2(\overline{M}') = \overline{M}'$. Hence we know the structure of $\overline{M}'$ by the main result of [33]. In particular, if V is a subgroup of R with $V \cong Q_8 * Q_8$, it follows that all involutions of $V - \langle z\rangle$ are conjugate in $N_{M'}(V)$ and hence in M. Since every involution of $R - \langle z\rangle$ lies in such a subgroup V, we see that all involutions of $R - \langle z\rangle$ are conjugate to z_1 in M. In particular, $t \sim z_1$. However, $z \sim z_1$, by Lemma 8.2, and so $z \sim t$, contrary to assumption.

Finally we establish Lemma 4.5(i).

LEMMA 8.8. *If G has only one conjugacy class of involutions, then all involutions of $R - \langle z\rangle$ are conjugate in $C_G(z)$.*

PROOF. We have $t^g = z_1$ for some g in G as G has only one conjugacy class of involutions. Let $S_1 \supseteq C_S(t)$ and $S_2 \supseteq C_S(z_1) = T$ be Sylow 2-subgroups of $C_G(t)$ and $C_G(z_1)$ respectively. We can clearly choose g so that $S_1^g = S_2$. Since $\Omega_1(S') = Z$, $\Omega_1(S_1') \cong \Omega_1(S_2') \cong Z_2 \times Z_2$. Since $z \in C_S(t)' \cap C_S(z_1)'$, it follows that $\Omega_1(S_1') = \langle t, z\rangle$ and $\Omega_1(S_2') = \langle z_1, z\rangle = Z$, whence $\langle t, z\rangle^g = Z$. Since $\langle z_1\rangle = Z(S_2)$, there exists s in S_2 such that $z^s = z_1 z$. Hence either g or gs maps t to z_1 and z to z. Thus t is conjugate to z_1 in M. Since also $z \in C_S(y)'$ for $y = tu$, u and uab, the same argument applies in these cases to yield that y is conjugate to z_1 in M. But z_1, t, tu, u, and uab are representatives of the conjugacy classes of involutions of $R - \langle z \rangle$ in S and so the lemma holds.

REFERENCES

The references below are intended as a list of papers that pertain to the existence and classification of simple groups of 2-ranks 3 and 4. Apart from the final category, which includes additional references required for the present paper, all papers which deal exclusively with groups of other 2-ranks have been deliberately omitted.

I. EXISTENCE OF SPORADIC GROUPS

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2. M. Hall, Jr., *A search for simple groups of order less than one million* (to appear).

3. D. Higman and C. Sims, *A simple group of order* 44,352,000, Math Z. **105** (1968), 110–113. MR **37** #2854.

4. G. Higman and J. McKay, *On Janko's simple group of order* 50,232,960, Bull. London Math. Soc. **1** (1969), 89–94. MR **40** #224.

5. Z. Janko, *A new finite simple group with abelian Sylow 2-subgroups and its characterization*, J. Algebra **3** (1966), 147–186. MR **33** #1359.

6. ———, *Some new simple groups of finite order*. I, Symposia Mathematica (INDAM, Rome, 1967/68), vol. 1, Academic Press, London, 1969, pp. 25–64. MR **39** #5686.

6*. R. Lyons, *Evidence for a new finite simple group* (to appear).

7. J. McLaughlin, *A simple group of order* 898,128,000, Proc. Sympos. Theory of Finite Groups, Benjamin, New York, 1969, pp. 109–111.

8. C. Sims, *On Lyons simple group of order* $2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$ (to appear).

II. CHARACTERIZATIONS OF SPORADIC GROUPS BY THEIR ORDERS

9. T. M. Gagen, *On groups with abelian Sylow 2-groups*, Math. Z. **90** (1965), 268–272. MR **33** #190.

10. M. Hall, Jr. and D. Wales, *The simple group of order* 604,800, J. Algebra **9** (1968), 417–450. MR **39** #1544.

11. D. Parrott and S. K. Wong, *On the Higman-Sims simple group of order* 44,352,000, Pacific J. Math. **32** (1970), 501–516. MR **41** #1867.

12. R. G. Stanton, *The Mathieu groups*, Canad. J. Math. **3** (1951), 164–174. MR **12**, 672.

13. S. Wong, *On a new finite non-abelian simple group of Janko*, Bull. Austral. Math. Soc. **1** (1969), 59–79.

III. CHARACTERIZATIONS OF SIMPLE GROUPS BY THE CENTRALIZERS OF THEIR INVOLUTIONS

14. P. Fong, *A characterization of the finite simple groups* $PSp(4, q)$, $G_2(q)$, $D_4^2(q)$. II, Nagoya Math. J. **39** (1970), 39–79.

15. P. Fong and W. Wong, *A characterization of the finite simple groups* $PSp(4,q)$, $G_2(q)$, $D_4^2(q)$. I, Nagoya Math. J. **36** (1969), 143–184. MR **41** #326.

16. M. Harris, *A characterization of the simple groups* PSp $(4,q)$, *q odd* (to appear).
17. ———, *A characterization of the simple groups* PSp $(4,q)$, $G_2(q)$, $D_4^2(q)$, *q odd* (to appear).
18. D. Held, *A characterization of the alternating groups of degrees eight and nine*, J. Algebra **7** (1967), 218–237. MR **36** #1530.
19. ———, *A characterization of some multiply transitive permutation groups.* I, Illinois J. Math. **13** (1969), 224–240. MR **39** #304.
20. T. Kondo, *A characterization of the alternating group of degree eleven*, Illinois J. Math. **13** (1969), 528–541. MR **40** #225.
21. Z. Janko, *A characterization of the Mathieu simple groups.* I, II, J. Algebra **9** (1968), 1–19, 20–41. MR **37** #5284.
22. Z. Janko and S. Wong, *A characterization of the Higman-Sims simple group*, J. Algebra **13** (1969), 517–534. MR **41** #5486.
23. ———, *A characterization of the McLaughlin simple group* (to appear).
24. K. Phan, *A characterization of four-dimensional unimodular group*, J. Alegbra **15** (1970), 252–279. MR **41** #3622.
25. ———, *A characterization of simple groups $U_4(q)$, q odd* (to appear).
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27. ———, *A characterization of the finite simple groups* $\mathrm{PSp}_6(q)$, *q odd*, J. Algebra **12** (1969), 494–524. MR **39** #2868.

IV. Characterizations of simple groups by their Sylow 2-subgroups

28. R. Brauer, *A characterization of* Sz(8) (unpublished).
29. R. Brauer and P. Fong, *A characterization of the Mathieu group* $\mathfrak{M}_{12}$, Trans. Amer. Math. Soc. **122** (1966), 18–47. MR **34** #7631.
30. D. Gorenstein and K. Harada, *A characterization of Janko's two new simple groups*, J. Univ. Tokyo **16** (1970), 331–406.
31. ———, *On finite groups with Sylow 2-subgroups of Type A_n, $n=8$, 9, 10, 11,* Math Z. **111** (1970), 207–238.
32. ———, *On finite groups with Sylow 2-subgroups of type $\hat{A}_n$, $n=8$, 9, 10, 11,* J. Algebra (to appear).
33. ———, *Finite groups whose Sylow 2-subgroups are the direct product of two dihedral groups*, Ann. of Math. (to appear).
34. ———, *Finite groups with Sylow 2-subgroups of type* PSp $(4, q)$, *q odd*, J. Math. Soc. Japan (to appear).

V. Simple groups with abelian Sylow 2-subgroups

35. H. Bender, *On groups with abelian Sylow 2-subgroups*, Math. Z. **111**(1970), 164–176.
36. Z. Janko and J. Thompson, *On a class of finite simple groups of Ree*, J. Algebra **4** (1966), 274–292. MR **34** #1386.
37. J. Thompson, *Toward a characterization of $E_2^*(q)$*, J. Algebra **7** (1967), 406–414. MR **36** #6496.
38. H. Ward, *On Ree's series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89. MR **33** #5752.
39. J. H. Walter, *The characterization of finite groups with abelian Sylow 2-subgroups*, Ann. of Math. (2) **89** (1969), 405–514. MR **40** #2749.

VI. General classification theorems

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41. ———, *Finite groups having a strongly embedded subgroup* (to appear).

42. Z. Janko, *Nonsolvable finite groups all of whose 2-local subgroups are solvable* (to appear).

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44. A. MacWilliams, *On 2-groups with no normal abelian subgroups of rank* 3 *and their occurrence as Sylow* 2-*subgroups of finite simple groups*, Trans. Amer. Math. Soc. **150** (1970), 345–408.

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VII. General classification methods

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VIII. Additional references

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56. ———, *Finite simple groups of* 2-*rank two*, Scripta Math. (to appear).

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