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INTEGRABILITY CONDITIONS FOR $\Delta u = k - Ke^{\alpha u}$ WITH APPLICATIONS TO RIEMANNIAN GEOMETRY

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1. In this note we announce some integrability conditions for the equation $\Delta u = k - Ke^{\alpha u}$ on compact orientable Riemannian 2-manifolds (where Δ is the Laplacian), and we give some applications to problems in Riemannian geometry. Further results and details will appear elsewhere. We begin with a description of the geometry problem which led us to a study of the above equation. M, throughout, will denote a compact, connected, oriented, 2-dimensional manifold.

Problem. What are necessary and sufficient conditions on a sufficiently smooth (we shall restrict ourselves to C^∞ data here) function K on M for K to be the Gaussian curvature of some Riemannian metric on M?

If K is the Gaussian curvature of a Riemannian metric g with volume form ω on M, the only known global condition which K must satisfy is the Gauss-Bonnet formula

$$\int_M K\omega = 2\pi\chi(M), \tag{1}$$

where $\chi(M)$ is the Euler characteristic of M. Here $K\omega$ is called the curvature form of the metric g. One can rephrase the above question in terms of curvature forms, and in this case it turns out [9] that the condition $\int_M \Omega = 2\pi\chi(M)$ is not only necessary but also is sufficient for a two form Ω to be the curvature form of a Riemannian metric on M. As for the Gaussian curvature functions themselves, (1) seems only to impose certain sign requirements depending on the genus of M. Specifically, it seems natural to expect that a necessary and sufficient condition for a smooth function K to be the Gaussian curvature of a Riemannian metric on M is

(i) that K be positive somewhere if genus$(M)=0$,
(ii) that K change sign, if not identically 0, if genus$(M)=1$,
(iii) that K be negative somewhere if genus$(M)>1$.

As a special case of (i), H. Gluck has recently shown [3] that K is a Gaussian curvature if K is strictly positive. His approach is to

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show that by composing K with a diffeomorphism one can satisfy the integrability conditions for the Minkowski problem. This approach, however, is limited to the case of genus 0 and strictly positive K.

Let g be a given Riemannian metric on M with Gaussian curvature k. We attack the above problem by trying to realize K (or $K \circ \phi$ where ϕ is an arbitrary diffeomorphism of M) as the curvature of a metric $\bar{g}$ pointwise conformal to g, that is, of the form $\bar{g} = e^{2u}g$ for some C^∞ function u on M. This approach has the advantage that, in principle at least, it is applicable to all cases (i)–(iii) and it leads directly to the specific partial differential equation for u,

$$\Delta u = k - Ke^{2u}, \tag{2}$$

where Δ is the Laplacian of the metric g.

In this form our problem is related to a question posed by L. Nirenberg who asked, for a given smooth strictly positive function K on S^2, whether or not there is a compact strictly convex surface Σ in E^3 and a conformal diffeomorphism $\phi : \Sigma \to S^2$ such that $K \circ \phi$ is the Gaussian curvature of Σ. This reduces directly to the question of the existence of solutions of $\Delta u = 1 - Ke^{2u}$ on S^2 for a given strictly positive K. It has been shown by D. Koutroufiotis [6] that this equation does have solutions for symmetric functions K on S^2 sufficiently close to the function 1. However, it is a consequence of one of our integrability conditions (see §3 below) that there are K arbitrarily close to 1 for which this equation has no solutions.

Melvyn Berger pointed out to us some work [1] that he had done on equation (2) by variational methods, and using these methods he has made some progress [2] on our questions (ii) and (iii), answering (iii) affirmatively for strictly negative K and providing a partial solution for (ii), a complete solution for which we announce below.

2. In this section M is a compact, connected, oriented, Riemannian 2-manifold, with Δ the associated Laplace operator and ω the volume form. Let f be a C^∞ function on M with $\int_M f\omega = 0$. In this situation we have a necessary and sufficient condition on h for there to exist a solution of $\Delta u = f + he^{\alpha u}$ for α a positive real constant.

THEOREM 1. *We consider the equation $\Delta u = f + he^{\alpha u}$ under the assumptions that $\int_M f\omega = 0$ and $\alpha > 0$. If $h \equiv 0$, the equation has a solution. If h is not identically zero, then a necessary and sufficient condition for there to exist a solution is that h take on both positive and negative values, and that $\int_M he^{\alpha v}\omega > 0$, where v is a solution of $\Delta v = f$.*

The proof uses a variational argument together with an extension of the Trudinger Inequality [8], [7], [5] to manifolds. In the case of the torus, this gives necessary and sufficient conditions for curvature functions to be related by a conformal change of metric. As an application of Theorem 1 we prove

THEOREM 2. *A necessary and sufficient condition for a smooth function K to be the Gaussian curvature of some Riemannian metric on the torus is that K change sign if not identically* 0.

A more subtle consequence of Theorem 1 is the following result, which one might expect since there is no Gauss-Bonnet theorem for the plane E^2.

THEOREM 3. *Each C^∞ function on the plane E^2 is the Gaussian curvature of some Riemannian on E^2.*

3. In this section we consider the equation $\Delta u = 1 - Ke^{2u}$ on the 2-sphere S^2, where Δ is the Laplacian of the standard metric. Our main result is the following integrability condition, which shows, among other things, that there are functions K on S^2 which are known to be curvature functions but which cannot be realized by a conformal change of the standard metric.

THEOREM 4. *A necessary condition on K for there to exist a solution of $\Delta u = 1 - Ke^{2u}$ on S^2 is that*

$$\int_{S^2} (e^{2u} \nabla K \cdot \nabla F)\omega = 0 \tag{3}$$

for all spherical harmonics F of degree 1. (*Here ∇ denotes the gradient on S^2.*)

This necessary condition can easily be generalized to cover the equation $\Delta u = k - Ke^{2u}$ with $\int_{S^2} k\omega = 4\pi$.

It follows immediately, for example, that $\Delta u = 1 - Ke^{2u}$ has no solutions if K is a spherical harmonic of degree 1 since in this case the integral in (3) is necessarily positive for $K = F$. Since the integral in (3) is unchanged by adding constants to K, one can easily construct examples of strictly positive C^∞ functions on S^2, for example $2 + \cos\phi$ (we use spherical coordinates $z = \cos\phi$, $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$) for which the equation has no solutions, thereby answering Nirenberg's question negatively. If one takes the special case in which $F = \cos\phi$, then (3), in spherical coordinates, becomes

$$\int_0^\pi \left\{ \int_0^{2\pi} e^{2u} K_\phi \sin^2\phi \, d\theta \right\} d\phi = 0.$$

It was this form of (3) to which we were first led by our observation of the nonexistence of rotationally symmetric (function of ϕ alone) solutions for $\Delta u = 1 - Ke^{2u}$ given rotationally symmetric data K (see [4]).

It appears that (3) poses no a priori constraint if one allows the modification of K by a diffeomorphism ϕ of S^2. Thus it is possible that for each K which is positive somewhere on S^2 there is a diffeomorphism ϕ of S^2 such that $\Delta u = 1 - (K \circ \phi)e^{2u}$ has a solution. If this be the case, then $K \circ \phi$ and hence K would be Gaussian curvatures of Riemannian metrics on S^2, thereby answering (i) affirmatively.

4. The equation in §3 becomes much more interesting if we free it from the geometric case of exponent 2 in e^{2u} and consider the equation $\Delta u = f + he^{\alpha u}$ on an arbitrary compact, connected, oriented Riemannian 2-manifold M, under the assumptions that $\int_M f\omega > 0$ and that h be negative somewhere. In this situation we have the following theorem, which has been observed in a special case by Berger and Moser.

THEOREM 5. *Suppose that* $(\mathrm{vol}(M))^{-1}\int_M f\omega = c > 0$, *that h is negative somewhere on M, and that α is a positive real constant. Then there exists* $\beta > 0$ *such that for* $0 < c\alpha < \beta$, *the equation* $\Delta u = f + he^{\alpha u}$ *always has a solution.*

As in the case of Theorem 1 the proof here uses a variational argument together with the Trudinger Inequality on manifolds. Using his sharp version of the Trudinger inequality for S^2 [7], Moser has shown that one can take $\beta = 2$ on S^2. Our Theorem 4 shows that 2 is the best possible value for β on S^2. One of the interesting phenomena here is the different nature of the constraints at the extremes of the range $0 < c\alpha < \beta$. At the value $c = 0$, corresponding to our Theorem 1, where we have a necessary and sufficient condition, the constraint is an integral inequality on h plus the requirement that h take on both positive and negative values. On S^2, for $c\alpha = 2$, the only known constraint so far is the "Gauss-Bonnet theorem" and an integral identity involving the derivatives of h. The fact that 2 is the best possible value of β for S^2 is intimately tied to the fact that 2 is the lowest nonzero eigenvalue of $(-\Delta)$. We have no information on S^2 for the range $2 < c\alpha < 6$. But again at 6, which is the next eigenvalue of $(-\Delta)$ we have a constraint analogous to (3) showing that there are rotationally symmetric K, for example $K = 3\cos^2\phi - 1$, for which $\Delta u = 1 - Ke^{6u}$ has no rotationally symmetric solutions.

ADDED IN PROOF. It follows from a strengthened version of Theorem 1 that a necessary and sufficient condition for a smooth function K on the Klein bottle to be the Gaussian curvature of some metric is that K change sign if not identically zero.

Recently, Moser has shown that (2) has antipodally symmetric solutions on S^2 if $k \equiv 1$ and if K is an antipodally symmetric function which is positive somewhere. From this it follows that the condition of being positive somewhere characterizes curvature functions on the real projective plane.

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