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A REGULARITY THEOREM FOR THE FIRST VARIATION OF THE AREA INTEGRAND[1]

BY WILLIAM K. ALLARD

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Suppose R^n is n-dimensional Euclidean space, k is an integer and $2 \leqq k < n$. We will state here a regularity theorem for k-dimensional varifolds in R^n which satisfy inequalities involving the first variation of their weighted k-dimensional area. Naturally formulated analogues of our results hold in any Riemannian manifold.

Though quite simple, the concept of a varifold is unfamiliar to the general mathematical public. In order to suggest the nature of our results in a context familiar to geometers, we first state two corollaries to our regularity theorem in which the varifolds under consideration are submanifolds of R^n. Let $M_k(R^n)$ be the set of continuously differentiable k-dimensional submanifolds of R^n of locally finite k-dimensional area; whenever $M \in M_k(R^n)$, let $\|M\|$ be the Radon measure on R^n which assigns to each open subset of R^n the k-dimensional area of its intersection with M. Whenever $a \in R^n$ and $0 < r < \infty$, let $U(a, r) = R^n \cap \{x : |x - a| < r\}$ and let S^{n-1} be the boundary of $U(0, 1)$. Let $\alpha(k)$ be the k-dimensional area of the ball of radius 1 centered at 0 in R^k.

COROLLARY. *Corresponding to each p with $k < p < \infty$ and each ϵ with $0 < \epsilon < 1$ there is $\eta > 0$ with the following property:*

Suppose

(a) *$M \in M_k(R^n)$ and $0 \in M$;*

(b) *$\|M\| U(0, 1) \leqq (1 + \eta)\alpha(k)$;*

(c) *M is smooth, M is closed relative to $U(0, 1)$ and*

$$k\left(\int_{U(0,1)} |H(x)|^p \, d\|M\|x\right)^{1/p} \leqq \eta,$$

where H is the mean curvature vector of M.

Then there are a linear isometry $\theta: R^n \to R^n$ and smooth functions $f_j: R^k \to R$, $j = 1, \cdots, n-k$, such that

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$$U(0, 1-\epsilon) \cap \theta(M) = U(0, 1-\epsilon) \cap \{x: x_{k+j} = f_j(x_1, \cdots, x_k),\ j = 1, \cdots, n-k\}$$

and such that

$$| D_i f_j(y) - D_i f_j(z) | \leqq \epsilon | y - z |^{1-k/p}$$

whenever $y, z \in \boldsymbol{R}^k$ *and* $i = 1, \cdots, k, j = 1, \cdots, n-k$.

Our next corollary is a rigidity theorem.

COROLLARY. *There is a positive number η with the following property: Suppose L is a smooth closed $(k-1)$-dimensional submanifold of S^{n-1}, L has zero mean curvature relative to S^{n-1} and the $(k-1)$-dimensional area of L does not exceed $(1+\eta)k\alpha(k)$. Then there is a k-dimensional linear subspace T of $\boldsymbol{R}^n$ such that $L = T \cap S^{n-1}$.*

We assert that the conclusions of the first corollary hold when (a), (b), (c) are naturally reformulated to apply to a class of objects far more general than the submanifolds of $\boldsymbol{R}^n$. This reformulation will imply a statement which includes the second corollary. Regularity theorems have been obtained for flat chain solutions to minimum problems defined by parametric elliptic integrands; see [2]. Although we study here only the area integrand, which is the simplest parametric integrand, we assume in no way that the surfaces considered minimize area; we do not even assume that the first variation of their area is zero, only that this first variation is summable to a power greater than the dimension of the surface.

Following Almgren (see [1]), we define a *k-dimensional varifold in $\boldsymbol{R}^n$* to be a Radon measure on $\boldsymbol{R}^n \times \boldsymbol{G}(n, k)$ where $\boldsymbol{G}(n, k)$ is the Grassmann manifold of k-dimensional linear subspaces of $\boldsymbol{R}^n$. Let $\boldsymbol{V}_k(\boldsymbol{R}^n)$ be the weakly topologized space of these varifolds. When $V \in \boldsymbol{V}_k(\boldsymbol{R}^n)$, let $\|V\|(A) = V(A \times \boldsymbol{G}(n, k))$ for $A \subset \boldsymbol{R}^n$; $\|V\|$ is called the *weight* of V. Whenever $M \in \boldsymbol{M}_k(\boldsymbol{R}^n)$, let

$$v(M)(B) = \|M\| \{x: (x, \operatorname{Tan}(M, x)) \in B\}, \qquad B \subset \boldsymbol{R}^n \times \boldsymbol{G}(n, k),$$

where $\operatorname{Tan}(M, x) \in \boldsymbol{G}(n, k)$ is the tangent space to M at $x \in M$; clearly, $v(M) \in \boldsymbol{V}_k(\boldsymbol{R}^n)$ and $\|v(M)\| = \|M\|$. Given $M_1, M_2, \cdots \in \boldsymbol{M}_k(\boldsymbol{R}^n)$ positive real numbers $c_1, c_2, \cdots$ and a Borel subset B of $\boldsymbol{R}^n$ such that $\sum_{i=1}^{\infty} c_i \|M_i\| (B \cap K) < \infty$ for every compact subset K of $\boldsymbol{R}^n$, we see that the restriction of $\sum_{i=1}^{\infty} c_i v(M_i)$ to $B \times \boldsymbol{G}(n, k)$ is a member of $\boldsymbol{V}_k(\boldsymbol{R}^n)$. Such a varifold is called *rectifiable*; if the c_i may be taken to be positive integers, it is called *integral*. The class of integral varifolds includes all classical geometric objects such as real analytic sets and continuously differentiable curvilinear complexes, as well as objects of more general singularity structure.

Suppose $F:\mathbf{R}^n\to\mathbf{R}^n$ is a univalent continuously differentiable mapping such that at each point $x\in\mathbf{R}^n$ the differential $DF(x)$ of F at x is nonsingular. For each $M\in\mathbf{M}_k(\mathbf{R}^n)$, let $F_\# M=\{F(x):x\in M\}\in\mathbf{M}_k(\mathbf{R}^n)$. We now extend $F_\#$ to $\mathbf{V}_k(\mathbf{R}^n)$. At each $(x, S)\in\mathbf{R}^n\times\mathbf{G}(n, k)$, let $J_kF(x, S)$ be the k-dimensional area of $DF(x)(S\cap\mathbf{U}(0, 1))$ divided by $\boldsymbol{\alpha}(k)$; for each $V\in\mathbf{V}_k(\mathbf{R}^n)$, let $F_\# V\in\mathbf{V}_k(\mathbf{R}^n)$ be such that

$$F_\# V(B) = \int_{\{(x,S):(F(x),DF(x)(S))\in B\}} J_kF(x, S)\, dV(x, S)$$

for each Borel subset B of $\mathbf{R}^n\times\mathbf{G}(n, k)$. It is then elementary that the diagram

$$\begin{array}{ccc} \mathbf{V}_k(\mathbf{R}^n) & \xrightarrow{F_\#} & \mathbf{V}_k(\mathbf{R}^n) \\ \mathbf{v}\uparrow & & \mathbf{v}\uparrow \\ \mathbf{M}_k(\mathbf{R}^n) & \xrightarrow{F_\#} & \mathbf{M}_k(\mathbf{R}^n) \end{array}$$

is commutative. Now let $\mathfrak{X}(\mathbf{R}^n)$ be the vector space of smooth functions $g:\mathbf{R}^n\to\mathbf{R}^n$ with compact support. For each $V\in\mathbf{V}_k(\mathbf{R}^n)$ we let

$$\delta V:\mathfrak{X}(\mathbf{R}^n)\to\mathbf{R}$$

be the linear functional which has at $g\in\mathfrak{X}(\mathbf{R}^n)$ the value

$$d/dt\|h_{t\#}V\|(U)\,|_{t=0},$$

where $h_t(x)=x+tg(x)$ for $(t, x)\in\mathbf{R}\times\mathbf{R}^n$ and U is a bounded open subset of $\mathbf{R}^n$ containing the support of g. As is well known,

$$\delta\mathbf{v}(M)(g) = -k\int g(x)\cdot H(x)\, d\|M\|x$$

whenever $M\in\mathbf{M}_k(\mathbf{R}^n)$, M is smooth, H is the mean curvature vector of M, $g\in\mathfrak{X}(\mathbf{R}^n)$ and the support of g meets M in a closed set. Thus δV is in some sense the mean curvature of V.

Whenever $V\in\mathbf{V}_k(\mathbf{R}^n)$, $1<p<\infty$, and G is an open subset of $\mathbf{R}^n$, we let $\boldsymbol{\nu}(V; p, G)$ be the supremum of the set of numbers $\delta V(g)$ corresponding to those $g\in\mathfrak{X}(\mathbf{R}^n)$ such that support $g\subset G$ and $(\int|g(x)|^q\, d\|V\|x)^{1/q}\leq 1$, where $q=p/(p-1)$. For example, if $M\in\mathbf{M}_k(\mathbf{R}^n)$, M is smooth, M is closed relative to G and H is the mean curvature vector of M, it is easy to see that

$$\boldsymbol{\nu}(\mathbf{v}(M); p, G) = k\left(\int_G |H(x)|^p\, d\|M\|x\right)^{1/p}.$$

If $\delta V(g)=0$ for all $g\in\mathfrak{X}(\boldsymbol{R}^n)$ with support $g\subset G$, we say *V is stationary in G.*

We now state our main result.

REGULARITY THEOREM. *Corresponding to each p with $k<p<\infty$ and each ϵ with $0<\epsilon<1$ there is $\eta>0$ with the following property:*

Suppose

(a) $V\in\boldsymbol{V}_k(\boldsymbol{R}^n)$, $0\in$support $\|V\|$ *and*

$$\limsup_{r\to 0}\frac{\|V\|\boldsymbol{U}(x,r)}{\alpha(k)r^k}\geqq 1 \qquad \text{for } \|V\| \text{ almost all } x\in\boldsymbol{U}(0,1);$$

(b) $\|V\|\boldsymbol{U}(0,1)\leqq(1+\eta)\alpha(k)$;

(c) $\boldsymbol{\nu}(V;p,\boldsymbol{U}(0,1))\leqq\eta$.

Then there are a linear isometry $\theta:R^n\to R^n$ and continuously differentiable functions $f_j:\boldsymbol{R}^k\to\boldsymbol{R}$, $j=1,\cdots,n-k$, such that

$$\begin{aligned}&\boldsymbol{U}(0,1-\epsilon)\cap\theta(\text{support }\|V\|)\\&\quad=\boldsymbol{U}(0,1-\epsilon)\cap\{x:x_{k+j}=f_j(x_1,\cdots,x_k),j=1,\cdots,n-k\}\end{aligned}$$

and such that

$$|D_if_j(y)-D_if_j(z)|\leqq\epsilon|y-z|^{1-k/p}$$

whenever $y,z\in\boldsymbol{R}^k$ and $i=1,\cdots,k$, $j=1,\cdots,n-k$. Moreover,

$$\Theta^k(\|V\|,x)=\lim_{r\to 0}\frac{\|V\|\boldsymbol{U}(x,r)}{\alpha(k)r^k}$$

is a real number for all $x\in\boldsymbol{U}(0,1)$ and, for V almost all $(x,S)\in\boldsymbol{U}(0,1-\epsilon)\times\boldsymbol{G}(n,k)$,

$$S=\text{Tan}(\text{support }\|V\|,x).$$

The theorem is proved using geometric measure theoretic techniques, some of which appear in [**1**] and [**3**] and are known to workers in this field, together with a variant of an argument used by Almgren in [**2**], to exploit well-known *a priori* estimates for the Laplacian. To extend the theorem to Riemannian manifolds, one uses Nash's isometric imbedding theorem (see [**4**]) and an elementary computation involving mean curvature. This regularity theorem is the main theorem in a paper which will study other aspects of the functional δV as well.

Suppose $V\in\boldsymbol{V}_k(\boldsymbol{R}^n)$, p is as in the Regularity Theorem, G is an open subset of $\boldsymbol{R}^n$, $\boldsymbol{\nu}(V;p,G)<\infty$, $0<d<\infty$, and

$$\limsup_{r\to 0}\|V\|\boldsymbol{U}(x,r)/\alpha(k)r^k\geqq d$$

for $\|V\|$ almost all $x \in G$.[2] Let S, the singular set of V in G, be the set of points x in $G \cap \text{support } \|V\|$ for which V does *not* have near x a continuously differentiable structure as in the conclusion of the Regularity Theorem. It can be shown that S is closed relative to G and has no interior; it is not true in general that $\|V\|(S) = 0$.[3] *Does the hypothesis that V is stationary in G imply that $\|V\|(S) = 0$?* The resolution of this question is an outstanding problem in the field.

Using topological arguments similar to those of Morse Theory, Almgren has proved the following theorem: *Suppose M is a compact Riemannian manifold and k is a positive integer less than the dimension of M. There is a nonzero k-dimensional integral varifold which is stationary in M.* See [1]. The regularity question for such varifolds was the main impetus for this work. Our theorem together with Morrey's work on higher differentiability (see [3]) implies that an open dense subset of the support of such a varifold is a smooth k-dimensional submanifold of M with zero mean curvature.

In the course of this work I have benefited from conversations with H. Federer and E. Bombieri; several consultations I have had with F. J. Almgren, Jr. were indispensable, and it was at his suggestion that I began this work.

References

1. F. J. Almgren, Jr., *The theory of varifolds*, Mimeographed notes, Princeton, 1965.

2. ———, *Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure*, Ann. of Math. (2) **87** (1968), 321–391. MR **37** #837.

3. H. Federer, *Geometric measure theory*, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, New York, 1969. MR **41** #1976.

4. J. F. Nash, Jr., *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63. MR **17**, 782.

PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

[2] It is a theorem that the restriction of V to $G \times G(n, k)$ is rectifiable.

[3] If the oscillation of $\Theta^k(\|V\|, \cdot)$ on support $\|V\|$ is less than ηd, it will be true that $\|V\|(S) = 0$.