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ON THE MEAN CURVATURE OF SUBMANIFOLDS OF EUCLIDEAN SPACE

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Let $x: M^n \to E^m$ be an immersion of an n-dimensional manifold M^n in a euclidean space E^m of dimension m $(m>n>1)$, and let ∇ and ∇' be the covariant differentiations of M^n and E^m, respectively. Let u and v be two tangent vector fields on M^n. Then the second fundamental form h is given by

$$\nabla'_u v = \nabla_u v + h(u, v). \tag{1}$$

If $\{e_1, \cdots, e_n\}$ is an orthonormal basis in the tangent space $T_p(M)$ at $p \in M^n$, then the mean curvature vector $H(p)$ at p is given by

$$H(p) = (1/n) \sum_{i=1}^{n} h(e_i, e_i). \tag{2}$$

Let $\langle \ , \ \rangle$ denote the scalar product of E^m. If there exists a function f on M such that $\langle h(u, v), H\rangle = f\langle u, v\rangle$ for all tangent vector fields u, v on M^n, then M^n is called a *pseudo-umbilical submanifold* of E^m. If the covariant derivative of H in E^m is tangent to $x(M^n)$ everywhere, then H is said to be parallel in the normal bundle. In [2], [3], the author proved that if M^n is closed, then the mean curvature vector H satisfies

$$\int_{M^n} \langle H, H\rangle^{n/2} dV \geqq c_n, \tag{3}$$

where dV denotes the volume element of M^n and c_n is the area of the unit n-sphere. The equality sign of (3) holds when and only when M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^m. It is interesting to know whether the inequality (3) can be improved for some special submanifolds of E^m.

The main purpose of this paper is to announce some results in this direction together with some results on pseudo-umbilical submanifolds. Details will appear elsewhere.

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By studying the behaviors of the mean curvature vector $\boldsymbol{H}$ we can prove

LEMMA 1. *The position vector field* $\boldsymbol{X}$ *of* M^n *in* E^m *is parallel to the mean curvature vector* $\boldsymbol{H}$ *when and only when* M^n *is either a minimal submanifold of* E^m *or a minimal submanifold of a hypersphere of* E^m *centered at the origin.*

By using Lemma 1, we can prove

PROPOSITION 1 (YANO-CHEN [5]). M^n *is a pseudo-umbilical submanifold of* E^m *such that the mean curvature vector* $\boldsymbol{H}$ *is parallel in the normal bundle when and only when* M^n *is either a minimal submanifold of* E^m *or a minimal submanifold of a hypersphere of* E^m.

If the codimension is equal to 2, then a pseudo-umbilical submanifold of E^m with constant mean curvature is always a pseudo-umbilical submanifold such that the mean curvature vector is parallel in the normal bundle. Hence we have

PROPOSITION 2. M^n *is a pseudo-umbilical submanifold of* E^{n+2} *with constant mean curvature when and only when* M^n *is either a minimal submanifold of* E^{n+2} *or a minimal hypersurface of a hypersphere of* E^{n+2}.

Let F be a field and $H_i(M^n, F)$ denote the ith cohomology group of M^n over the field F. Let $\beta(M^n) = \max\{\sum_{i=0}^{n} \dim H_i(M^n, F); F \text{ fields}\}$. Then, by verifying the properties of the length of second fundamental form h, we can prove

THEOREM I. *Let* M^n *be an n-dimensional closed manifold immersed in* E^m *with nonnegative scalar curvature. Then we have*

$$\int_{M^n} \langle \boldsymbol{H}, \boldsymbol{H}\rangle^{n/2}\, dV > a\beta(M^n), \tag{4}$$

where

$$\begin{aligned} a &= (4n^n)^{-1/2}c_n, && \textit{if } n \textit{ is even}, \\ &= (2n^n c_{m-n-1}c_{m+n-1})^{-1/2}(c_{2n})^{1/2}c_{m-1}, && \textit{if } n \textit{ is odd}. \end{aligned} \tag{5}$$

THEOREM II. *Let* M^2 *be a flat torus in* E^4. *Then we have*

$$\int_{M^2} \langle \boldsymbol{H}, \boldsymbol{H}\rangle dV \geqq 2\pi^2. \tag{6}$$

Then the equality sign of (6) *holds when and only when* M^2 *is a Clifford torus in* E^4.

PROOF (SKETCH). The proof of (6) follows from a direct computation of the first curvature of second kind, $\lambda_1(p)$, (for the definition, see [1]) and the relations between the mean curvature and λ_1. If the equality of (6) holds, then we can prove that M^2 is a minimal surface of a 3-sphere in E^4. From this we see that M^2 is a Clifford torus in E^4. The converse of this is trivial.

For each unit normal vector e to $x(M^n)$ at $x(p)$, let h_e be the linear transformation from the tangent space $T_p(M)$ into itself defined by

$$\langle h_e(u), v\rangle = \langle h(u, v), e\rangle \tag{7}$$

for all tangent vectors u, v at p. Let $K(p, e) = \det(h_e)$. Then $K(p, e)$ is called the Lipschitz-Killing curvature at (p, e). By deriving some integral formulas for the αth curvatures of first and second kinds (for the definitions, see [1]), we can prove

THEOREM III. *Let M^2 be an oriented closed surface in E^m. If M^2 is contained in a hypersphere of E^m, then M^2 is a pseudo-umbilical surface of E^m when and only when the Lipschitz-Killing curvature in the unit direction of the mean curvature vector H is maximal over the fibre of the unit normal bundle B_v, $B_v = \{(p, e) : p \in M^2$, e a unit normal vector in E^m at $x(p)\}$.*

THEOREM IV (ADDED IN PROOF). *The Veronese surface in E^5, the generalized Clifford tori in E^{n+2} and the n-sphere in E^{n+p} are the only closed pseudo-umbilical submanifolds M^n of E^{n+p} with mean curvature vector nowhere zero satisfying*

$$R \geqq \frac{n(p-1)\langle H, H\rangle}{2p-3}\left[(n-1)\left(\frac{2p-3}{p-1}\right)-1\right] \tag{8}$$

where R denotes the scalar curvature of M^n.

The proof of this theorem will appear in a forthcoming paper "Pseudo-umbilical submanifolds in a Riemannian manifold of constant curvature. II".

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