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FROBENIUS RECIPROCITY IN ERGODIC THEORY

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1. Introduction. The analogy between group actions and group
representations, which has guided algebraists' intuition since the turn
of the century, has been slow in influencing analysis. To be sure, the
analogy between "strict sense" and "wide sense" notions, first noticed
in probability, can be traced to be a natural analog of the preceding
analogy, and in fact has led to the discovery of a host of new results
(for example by Nelson[1] and Rota where the strict sense analogs of
well-known Hilbert-space theorems of Nagy and Naĭmark are worked
out). More recently, a conjecture of Rota (1962) regarding a strict-
sense analog of Schreiber's Theorem has been settled in the affirma-
tive by McCabe and Shields, using Ornstein's deep results relating
entropy to conjugacy. Despite this and much other work, however, a
systematic "spectral theory in the strict sense" for ergodic trans-

formations of measure spaces remains to be developed. The far-reaching ideas recently introduced by Mackey, and arrived at by analogies between transitive actions of finite groups and ergodic actions of locally compact groups, offer renewed promise for this program. The notion of virtual group, introduced by Mackey with the purpose of developing an "action" analog of the notion of induced representation, has already found a multitude of applications (Lange and Ramsay) and, we surmise, will substantially influence and deflect the development of ergodic theory.

Our present purpose is to strengthen and develop one of Mackey's constructions by placing it in the context of categorical algebra. It has been known for some time (see e.g. Lang) that the induced representation functor is the left adjoint of the restriction (to a subgroup) functor. Guided by this analogy, we are led to surmise that the construction of an induced action should be, in a suitable context, an adjoint functor. This turns out to be the case (Theorem 2). To establish it, a detailed detour into the treacherous waters of pointwise measure theory seems necessary. Some of the basic spadework has been done by Mackey, but a few notions need to be retouched. For this and other reasons, we have found it prudent to give all definitions, some of which—in particular that of virtual group—slightly differ from Mackey's.

The present work points to several further developments, notably: a "strict sense" analog of the imprimitivity theorem; a corresponding "right adjoint" to the functor R; and a development of the functor R and its left adjoint M, the "Mackey functor", along the lines of the theory of triples. We hope to consider them in forthcoming publications.

2. **Definitions.** Recall that a group is a small category with one object, where every morphism has an inverse. A *groupoid* is a small category where every morphism has an inverse.

We refer to Moore for the definitions of *Borel space, Borel map, countably generated Borel space, analytic Borel space, equivalent σ-finite measures, measure class* (denoted by **C**). Let T be a Borel map; if $T^{-1}(A)$ is a null set whenever A is a null set then the measure class is *invariant* under T. An *analytic Borel groupoid* is a groupoid, together with an analytic Borel space structure on the set of morphisms, and a measure class **C** (this induces a similar structure on the set U of objects of V by identifying an object with the corresponding identity morphism), such that:

(a) The domain D of composition of two morphisms is a Borel subset of $V \times V$, under the product Borel structure.

(b) Composition $(f, h) \to fh$ and inversion $f \to f^{-1}$ are Borel maps of $D(V)$ to V, and inversion leaves invariant the measure class $\mathbf{C}$.

Let d and r be the (Borel) maps of a morphism to its "domain" and "range"; then d induces a measure class $\bar{\mathbf{C}}$ on U, setting $\tilde{\mu}(A)$ $= \mu(d^{-1}(A))$ for μ in $\mathbf{C}$. Any μ in C satisfies $\mu = \int \mu_s \tilde{\mu}(ds)$, where μ_s is a measure on V living on $d^{-1}(s)$, and, for fixed Borel set A in V, the function $s \mapsto \mu_s(A)$ is Borel on U. The fibering is unique: if $\mu = \int \nu_s \tilde{\mu}(ds)$, then $\nu_s = \mu_s$ for $\bar{C}$—almost all s; furthermore, changing μ to an equivalent measure does not change the measure class of μ_s for $\bar{\mathbf{C}}$—almost all s; this leads to measure classes C_s on almost all fibers $d^{-1}(s)$.

(c) For $s \in U$ and $r(f) = s$, the map $h \mapsto hf$ carries $d^{-1}(s)$ biuniquely to $d^{-1}(d(f))$ and C_s to $C_s f$.

We require that $C_{r(f)} f = C_{d(f)}$ for all f with $r(f)$ and $d(f)$ in some *co-null* (complement of a null) Borel set U_0 of U. Say that $(V, \mathbf{C})$ is *ergodic* whenever every real Borel function ϕ on U such that $\phi(d(f))$ $= \phi(r(f))$ for almost all f in V is $\bar{\mathbf{C}}$ a.e. constant. A *virtual group* is an *ergodic analytic Borel groupoid*. In a virtual group $(V, \mathbf{C})$, let U_0 be a co-null Borel set of U. Taking all $f \in V$ s.t. both $d(f)$ and $r(f)$ are in U_0, we obtain another virtual group, the *inessential contraction* $V | U_0$. Again, say that two objects u and v are *equivalent* when $d(f) = u$ and $r(f) = v$ for some morphism f; if $A \subseteq U$, write $[A]$ for the *saturation* of A under this equivalence relation; note that $[A] = r(d^{-1}(A))$ and that $[A]$ is naturally an analytic Borel space if A is a Borel set.

3. Categories and functors. A *strict homomorphism* ψ between virtual groups $(V_1, \mathbf{C}_1)$ and $(V_2, \mathbf{C}_2)$ is a functor from V_1 to V_2 which is also a Borel map, and s.t. if $\bar{\psi}$ is the associated map of the objects U_1 of V_1 to the U_2 of V_2, then $\bar{\psi}^{-1}(A)$ is a $\bar{\mathbf{C}}_1$-null set for every saturated analytic $\bar{\mathbf{C}}_1$-null set A. A *homomorphism* of $(V_1, \mathbf{C}_1)$ to $(V_2, \mathbf{C}_2)$ is a functor which is a Borel map, and whose restriction to some inessential contraction of $(V_1, \mathbf{C}_1)$ is a strict homomorphism. Two homomorphisms ψ_1 and $\psi_2 : (V_1, \mathbf{C}_1) \to (V_2, \mathbf{C}_2)$ are *strictly similar* if $\theta(r(f)) \psi_1(f) = \psi_2(f) \theta(d(f))$ for all $f \in V_1$ and for some Borel map $\theta : U_1 \to V_2$ for which both sides are defined; ψ_1 and ψ_2 are *similar* if there is an inessential contraction of V_1 on which they are strictly similar. (*Note.* The composition of two homomorphisms is not necessarily a homomorphism!) Similarity is an equivalence relation, and similarity classes $[\phi]$ of homomorphisms ϕ are preserved under composition. Taking virtual groups as objects and similarity classes of homomorphisms as morphisms, one obtains a category. A locally compact separable group G is a virtual group when endowed with its Haar measure class. A virtual group V, together with a homomorphism $\pi : V \to G$, briefly V_π, will be called a *virtual group over* G. The *Mackey*

category **M**(*G*) of *G* has the virtual groups over *G* as objects, and as morphisms the similarity classes of homomorphisms which make the obvious triangle over *G* commute, namely, which commute with the action of [π]. The category **R**(*G*) *of ergodic actions* of *G* has as objects the transformation spaces *T* of *G*, namely:

(a) *T* is an analytic Borel space;

(b) the map $(t, x) \to tx$ of $T \times G \to T$ is Borel;

(c) *T* has a measure class **C** which is invariant under the set of Borel automorphisms $t \to tx$;

(d) (*T*, **C**) is ergodic: the only invariant Borel sets are null or co-null.

The morphisms of **R**(*G*) are equivalence classes of maps, as follows: T_1 and T_2 being objects, consider all Borel maps $\phi: T_1 \to T_2$ s.t.

(a) There is a co-null invariant analytic subset of T_1 on which ϕ is *G*-equivariant, that is, $\phi(tx) = \phi(t)x$.

(b) If *N* is a null set in T_2, so is $\phi^{-1}(N)$.

Two such maps ϕ and ψ are *equivalent* if there is a Borel map $\alpha: T_1 \to G$ such that $\phi(s)\alpha(s) = \psi(s)$ and $\alpha(sx) = x^{-1}\alpha(s)x$ for all *s* in some co-null invariant analytic subset of T_1. This equivalence is preserved under composition of maps; the equivalence classes are the morphisms of **R**(*G*). For a given ergodic action of *G* on *T*, give $T \times G$ the product Borel structure and product measure class and define $\pi: T \times G \to G$ as the projection. Defining $(s, x)(t, y) = (s, xy)$ whenever $sx = t$ gives $T \times G$ a groupoid structure whose objects are naturally identical with the set *T*. This construction maps the objects of **R**(*G*) into the objects of the Mackey category **M**(*G*).

THEOREM 1. *There is a functor* $R: \mathbf{R}(G) \to \mathbf{M}(G)$ *extending the above construction, which is faithful on objects and morphisms and whose image is a full subcategory of the Mackey category.*

The easy proof, following Mackey's techniques, is omitted.

4. **Main result.**

THEOREM 2. *The functor R has a left adjoint M.*

We call *M* the *Mackey functor*; a sketch of the proof follows, leaving fine measure-theoretic points to a detailed publication to follow. All homomorphisms and similarities are tacitly taken as *strict*. Given V_π, we first construct a transformation space *T* by generalizing a construction of Mackey, which is, in turn, patterned after a classical construction of induced representations. Take $U \times G$, where *U* is the object of *V*, and define a *G*-action by $(u, y)x = (u, x^{-1}y)$. Set (u, x)

$\sim(v, y)$ iff $(r(f), x) = (u, x)$ and $(d(f), x\pi(f)) = (v, y)$ for some $f \in V$. The G-action commutes with the equivalence $\sim$, so that the quotient $U \times G/\sim$ is a G-space (but unfortunately, not an analytic Borel G-space under the quotient Borel structure). It is however possible (using some results of Mackey) to replace $U \times G/\sim$ by a Borel space $M(V_\pi)$ having the same measure algebra, and the action of G is easily seen to be ergodic. This defines the functor M on the objects; next, we define it on morphisms. Given $[\phi]: V_1, \pi_1 \to V_2$, π_2 in $\mathbf{M}(G)$, the map $(u, x) \to (\phi(u), x\theta(u))$ of $U_1 \times G \to U_2 \times G$, where $\theta(r(f))\pi_2 \circ \phi(f) = \pi_1(f)\theta(d(f))$, factors through $\sim$ to give a map $U_1 \times G/\sim \to U_2 \times G/\sim$. If θ is changed to another θ_1 implementing the similarity, then $(\phi(u), x\theta(u))x\theta(u)\theta_1(u)^{-1}x^{-1} = (\phi(u), x\theta_1(u))$, and the map $(u, x) \to x\theta(u)\theta_1(u)^{-1}x^{-1}$ factors to give a map $\alpha: U_1 \times G/\sim \to G$ satisfying $\alpha(tx) = x^{-1}\alpha(t)x$. Thus, $(u, x) \to (\phi(u), x\theta(u))$ and $(u, x) \to (\phi(u), x\theta_1(u))$ factor to give G-equivariant maps $U_1 \times G/\sim \to U_2 \times G/\sim$ which belong to the same morphism class in $\mathbf{R}(G)$. Again, if $\gamma(r(f))\phi(f) = \phi_1(f)\gamma(d(f))$, then $u \to \theta(u)\pi_2 \circ \gamma(u)^{-1}$ implements the similarity between $\pi_2 \circ \phi_1$ and π_1 (recall $V_i: \pi_i \to G$). The maps $(u, x) \to (\phi_1(u), x\theta(u)\pi_2 \circ \gamma(u)^{-1})$ and $(u, x) \to (\phi(u), x\theta(u))$ yield the same map $U_1 \times G/\sim \to U_2 \times G/\sim$. Thus, the map of morphisms in $\mathbf{M}(G)$ to morphisms in $\mathbf{R}(G)$ is welldefined. It can be verified that the definition preserves composition of morphisms. Next, given $[\phi]: V_\pi \to R(T)$, we must associate a morphism $M(V_\pi) \to T$. Let $\bar\phi$ be the projection of ϕ onto T and β be the projection of $T \times G \to G$. For $\theta(r(f))\beta \circ \phi(f) = \pi(f)\theta(d(f))$, the map $(u, x) \to \bar\phi(u)\theta(u)^{-1}x^{-1}$ is constant on $\sim$-equivalence classes and G-equivariant, so that it induces a map $U \times G/\sim \to T$; it is again checked as above that changing θ or ϕ does not change the equivalence class of maps. Conversely, to a morphism $[\phi]: M(V_\pi) \to T$ we must associate a morphism $V_\pi \to R(T)$. Let ρ be the natural projection of $U \times G \to U \times G/\sim$, and set $\hat\phi(f) = (\phi \circ \rho(r(f), e), \pi(f))$, when e is the identity of G; one can show that $\hat\phi(fh) = \hat\phi(f)\hat\phi(h)$, that the required diagram commutes, and that changing to an equivalent map changes $\hat\phi$ only up to similarity. Lastly, one must verify that the two constructions are inverses and naturality of the whole construction. All this can be done with little difficulty.

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