BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 4, July 1971

NORMAL CONTROL PROBLEMS HAVE NO MINIMIZING STRICTLY ORIGINAL SOLUTIONS[1]

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Communicated by Victor Klee, February 12, 1971

ABSTRACT. We prove for a general optimal control problem that, in the absence of abnormal admissible extremals (solutions of a generalized Weierstrass E-condition), any control which is optimal in the set of original (ordinary) controls must also be optimal in the larger set of relaxed (measure-valued) controls.

1. We consider the model of an optimal control problem studied in [2]. This model was found applicable, among others, to unilateral control problems defined by ordinary differential and multidimensional integral equations [3], evasion problems [4], and conflicting control problems [5]. For the sake of completeness, we begin by restating the definition of this model. Let T and R be compact metric spaces and μ a positive and nonatomic Radon measure on T. We denote by $\mathrm{rpm}(R)$ the set of regular Borel probability measures on R endowed with the relative weak star topology of $C(R)^*$, by $\mathcal{R}$ the class of μ-measurable functions on T to R (*original control functions*), and by $\mathcal{S}$ the set of μ-measurable functions on T to $\mathrm{rpm}(R)$ (*relaxed control functions*). We embed R in $\mathrm{rpm}(R)$ and $\mathcal{R}$ in $\mathcal{S}$ by identifying each $r \in R$ with the Dirac measure at r. In turn, we view $\mathcal{S}$ as a subset of $L^1(T, C(R))^*$, and endow it with the relative weak star topology, by identifying each $\sigma \in \mathcal{S}$ with the functional $\phi \to \int \mu(dt) \int \phi(t)(r)\sigma(t)(dr)$.

Now let $\boldsymbol{R}$ be the real line, $\mathfrak{X}$ a real topological vector space, C a convex body in $\mathfrak{X}$, B a convex subset of a vector space (the *set of control parameters*), m a positive integer, $x = (x_0, x_1, x_2) : \mathcal{S} \times B \to \boldsymbol{R} \times \boldsymbol{R}^m \times \mathfrak{X}$ a given function, and

$$\mathcal{A}(\mathcal{U}) = \{(\sigma, b) \in \mathcal{U} \times B \mid x_1(\sigma, b) = 0,\ x_2(\sigma, b) \in C\} \qquad (\mathcal{U} \subset \mathcal{S}).$$

We say that $(\bar{\sigma}, \bar{b})$ is a *minimizing original* (respectively *relaxed*) *solution* if it yields a minimum of x_0 on $\mathcal{A}(\mathcal{R})$ (respectively on $\mathcal{A}(\mathcal{S})$). A minimizing original solution is a *minimizing strictly original solution* if it is not at the same time a minimizing relaxed solution. We set $Q = \mathcal{S} \times B$, denote by $\mathfrak{I}_{m+1}$ the simplex $\{(\theta^0, \cdots, \theta^m) \in \boldsymbol{R}^{m+1} \mid \theta^j \geqq 0,$

AMS 1970 *subject classifications.* Primary 49A25, 49B30; Secondary 49A50.

Key words and phrases. Model, optimal control problem, minimizing original solution, minimizing relaxed solution, abnormal extremal.

[1] Partly supported by N.A.S.A. grant NGR 22-011-020.

$\sum_{j=0}^{m}\theta^j \leqq 1\}$ and by $Dx(\tilde{q};\ q-\tilde{q})$ the directional derivative $\lim_{\alpha\to+0}\alpha^{-1}[x(\tilde{q}+\alpha(q-\tilde{q}))-x(\tilde{q})]$. For $\tilde{q}, q_0, q_1, \cdots, q_m \in Q$, we say that the function

$$\theta \to x\left(\tilde{q} + \sum_{j=0}^{m} \theta^j(q_j - \tilde{q})\right) : \mathfrak{J}_{m+1} \to \boldsymbol{R} \times \boldsymbol{R}^m \times \mathfrak{X}$$

is differentiable at 0 if it has a Fréchet derivative at 0 relative to $\mathfrak{J}_{m+1}$, i.e. if

$$\lim |\theta|^{-1}\left[x\left(\tilde{q} + \sum_{j=0}^{m}\theta^j(q_j - \tilde{q})\right) - x(q) - \sum_{j=0}^{m}\theta^j Dx(\tilde{q};\ q_j - \tilde{q})\right] = 0$$

in $\boldsymbol{R}\times\boldsymbol{R}^m\times\mathfrak{X}$ as $|\theta|\to 0$, $\theta\in\mathfrak{J}_{m+1}$.

Points $\tilde{q}=(\tilde{\sigma}, \tilde{b})\in \mathcal{S}\times B$ and $l=(l_0, l_1, l_2)\in[0, \infty)\times\boldsymbol{R}^m\times\mathfrak{X}^*$ define an *extremal* $(\tilde{q}, l)$, and $\tilde{q}$ is *extremal* if $\tilde{q}$ and l satisfy the generalized Weierstrass E-condition (maximum principle)

$$l \neq 0, \quad l(Dx(\tilde{q};\ q - \tilde{q})) \geqq 0 \quad (q \in Q) \quad \text{and} \quad l_2(x_2(\tilde{q})) \geqq l(c) \quad (c \in C).$$

An extremal $(\tilde{q}, l)$ is *admissible* if $\tilde{q}=(\tilde{\sigma}, \tilde{b})\in\mathfrak{A}(\mathcal{S})$; an extremal $(\tilde{q}, l) =(\tilde{q}, l_0, l_1, l_2)$ is *abnormal* if $l_0=0$. The optimal control problem is *normal* if there exist no abnormal admissible extremals.

THEOREM I. *Assume that, for each choice of $\tilde{q}, q_0, \cdots, q_m\in Q$, with $\tilde{q}=(\tilde{\sigma}, \tilde{b})$ and $q_i=(\sigma_i, b_i)$ $(i=0, 1, \cdots, m)$, the function*

$$(\sigma, \theta) \to x\left(\sigma, \tilde{b} + \sum_{j=0}^{m}\theta^j(b_j - \tilde{b})\right) : \mathcal{S} \times \mathfrak{J}_{m+1} \to \boldsymbol{R} \times \boldsymbol{R}^m \times \mathfrak{X}$$

is continuous and the function

$$\theta \to x\left(\tilde{q} + \sum_{j=0}^{m}\theta^j(q_j - \tilde{q})\right) : \mathfrak{J}_{m+1} \to \boldsymbol{R} \times \boldsymbol{R}^m \times \mathfrak{X}$$

is differentiable at 0. *If $(\bar{\rho}, \bar{b})$ is a minimizing strictly original solution then there exists an abnormal admissible extremal $(\sigma^\#, b^\#, 0, l_1, l_2)$ such that $x_0(\sigma^\#, b^\#) < x_0(\bar{\rho}, \bar{b})$.*

PROOF. Let $(\bar{\rho}, \bar{b})$ be a minimizing strictly original solution. We set

$$B' = B \times \boldsymbol{R}, \qquad \mathfrak{X}' = \mathfrak{X} \times \boldsymbol{R}, \qquad C' = C \times (-\infty, 0),$$

$$x_0'(\sigma, b') = \alpha, \qquad x_1'(\sigma, b') = x_1(\sigma, b),$$

$$x_2'(\sigma, b') = (x_2(\sigma, b), x_0(\sigma, b) - x_0(\bar{\rho}, \bar{b})) \quad (\sigma \in \mathcal{S},\ b' = (b, \alpha) \in B'),$$

$$x' = (x_0', x_1', x_2').$$

We denote by P the optimal control problem we are considering and by P' the problem obtained by replacing B, $\mathfrak{X}$, C and x with B', $\mathfrak{X}'$, C' and x', respectively. Since $(\bar{\rho}, \bar{b})$ does not minimize x_0 on $\mathfrak{A}(\mathcal{S})$, there exists $(\sigma^\#, b^\#) \in \mathfrak{A}(\mathcal{S})$ such that $x_0(\sigma^\#, b^\#) < x_0(\bar{\rho}, \bar{b})$. It follows that $x_1'(\sigma^\#, b^\#, 0) = 0$ and $x_2'(\sigma^\#, b^\#, 0) \in C'$. The argument of [**2**, 4.1, Proof of Theorem 2.2, p. 369], when applied to P' and $(\sigma^\#, b^\#, 0)$, shows that either (a) there exists $l = (l_0, l_1, l_2') \in [0, \infty) \times \boldsymbol{R}^m \times (\mathfrak{X} \times \boldsymbol{R})^*$ such that $(\sigma^\#, b^\#, 0, l_0, l_1, l_2')$ is an admissible extremal of P', or (b) there exists $(\rho_1, b_1, \alpha_1) \in \mathfrak{R} \times B'$ such that $x_1'(\rho_1, b_1, \alpha_1) = x_1(\rho_1, b_1) = 0$ and $x_2'(\rho_1, b_1, \alpha_1) \in C'$; hence $x_2(\rho_1, b_1) \in C$ and $x_0(\rho_1, b_1) < x_0(\bar{\rho}, \bar{b})$. The alternative (b) must be discarded because it conflicts with the assumption that $(\bar{\rho}, \bar{b})$ is a minimizing original solution. We set, in (a), $l_2' = (l_2, \lambda_0) \in \mathfrak{X}^* \times \boldsymbol{R}$ and $q^\# = (\sigma^\#, b^\#)$, and conclude that

$$l \neq 0, \qquad l_0\alpha + l_1 Dx_1(q^\#; q - q^\#) + l_2(Dx_2(q^\#; q - q^\#)) + \lambda_0 Dx_0(q^\#; q - q^\#) \geqq 0$$

$$(\alpha \in \boldsymbol{R},\ q = (\sigma, b) \in \mathcal{S} \times B)$$

and

$$l_2(x_2(q^\#)) + \lambda_0[x_0(q^\#) - x_0(\bar{\rho}, \bar{b})] \geqq l_2(c) + \lambda_0\alpha \quad (c \in C,\ \alpha \in (-\infty, 0)).$$

Since $x_0(q^\#) < x_0(\bar{\rho}, \bar{b})$, these relations imply that $\lambda_0 = l_0 = 0$ and show that $(\sigma^\#, b^\#, 0, l_1, l_2)$ is an abnormal admissible extremal of P. Q.E.D.

2. Theorem I can be applied, under certain conditions, to problems where the original control functions are not a priori restricted to a compact set (e.g. to a problem of Bolza when its admissible extremals have uniformly bounded derivatives). Examples can be given [**6**, p. 118] of simple problems that possess minimizing strictly original solutions but, in view of Theorem I, these problems cannot be normal. If we add (to those of Theorem I) the assumptions that $\mathfrak{A}(\mathcal{S})$ is nonempty and there exists a sequentially compact topology of B such that x is continuous on $\mathcal{S} \times B$ (or an appropriate subset) then, by [**2**, Theorems 2.1 and 2.2, pp. 362–363], there exists a minimizing relaxed solution and it is extremal. Thus, in normal problems, a minimizing original solution exists if and only if there exists an extremal point $(\bar{\rho}, \bar{b}) \in \mathfrak{A}(\mathfrak{R})$ that minimizes x_0 among all extremal $(\tilde{\sigma}, \bar{b}) \in \mathfrak{A}(\mathcal{S})$. This suggests that the most promising approach to a theory of minimizing original solutions will remain the one that led to the justification of the Dirichlet principle and that McShane [**1**] applied in 1940 to the Bolza problem (using Young's [**7**], [**8**] generalized curves as tools); namely, the investigation of conditions insuring that weak solutions of the problem (such as minimizing gen-

eralized curves or minimizing relaxed solutions) are also "classical" solutions.

We expect to publish elsewhere extensions of Theorem I with somewhat weaker hypotheses and with original control functions restricted by the condition $\rho(t) \in R^{\#}(t)$ μ-a.e., where $R^{\#}(\cdot)$ is a given μ-measurable set-valued mapping. We shall also demonstrate the applicability of the model to functional-integral equations in $C(T, \boldsymbol{R}^n)$ and $L^p(T, \boldsymbol{R}^n)$.

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