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SEMIAPOSYNDETIC NONSEPARATING PLANE CONTINUA ARE ARCWISE CONNECTED

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It is known that if H is an aposyndetic nonseparating plane continuum, then H is locally connected. This follows from a result of Jones' [2, Theorem 10] that if p is a point of a plane continuum H and H is aposyndetic at p, then the union of H and all but finitely many of its complementary domains is connected im kleinen at p.[2] As a corollary of these results, each aposyndetic nonseparating plane continuum is arcwise connected. Closely related to the notion of an aposyndetic continuum is that of a semiaposyndetic continuum, studied in [1]. A continuum M is *semiaposyndetic* if for each pair of distinct points x and y of M, there exists a subcontinuum F of M such that the sets $M-F$ and the interior of F relative to M each contain a point of $\{x, y\}$. Note that a nonseparating semiaposyndetic plane continuum may fail to be locally connected. The main theorem of this paper is that each semiaposyndetic nonseparating plane continuum is arcwise connected. A complete proof of this result will appear elsewhere. For definitions of unfamiliar terms and phrases see [4].

Throughout this paper S is the plane and d is the Euclidean metric for S.

DEFINITION. Let E be an arc-segment (open arc) in S with endpoints a and b, D be a disk in a continuum M in S, and ϵ be a positive real number. The arc-segment E is said to be *ϵ-spanned* by D in M if $\{a, b\}$ is a subset of D and for each point x in a bounded complementary domain of $D \cup E$, either $d(x, E) < \epsilon$ or x belongs to M.

DEFINITION. A point y of a continuum M *cuts* x from z in M if x, y

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[2] A continuum H is said to be *aposyndetic* at a point p of H with respect to a point q of $H-\{p\}$ if there exist an open set U and a continuum L in H such that $p \in U \subset L \subset H-\{q\}$. A continuum H is said to be *aposyndetic* at a point p if for each point q of $H-\{p\}$, H is aposyndetic at p with respect to q. If H is aposyndetic at each of its points, then H is said to be *aposyndetic* (Jones).

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and z are distinct points of M and y belongs to each subcontinuum of M which contains $\{x, z\}$.

The following lemmas are necessary preliminaries.

LEMMA 1. *If an arc-segment E in S of diameter less than ϵ with endpoints a and b is ϵ-spanned by a disk D in M (a continuum in S), then there exists an arc-segment $M(E)$ in M with endpoints a and b such that for each point x of $M(E)$, $d(x, E) \leqq 2\epsilon$.*

LEMMA 2. *If M is a semiaposyndetic metric continuum and x, y and z are points of M such that y cuts x from z in M, then z does not cut x from y in M.*

THEOREM. *If M is a semiaposyndetic continuum in S which does not separate S, then M is arcwise connected.*

PROOF. (SKETCH). Let p and q be distinct points of M. According to Jones' cyclic connectivity theorem [3], if no point cuts p from q in M, then p and q belong to a simple closed curve in M and are therefore the extremities of an arc lying in M. Suppose there exists a point which cuts p from q in M. Let K be the closed subset of M consisting of p, q and all points x such that x cuts p from q in M. Define the binary relation R on K as follows. For distinct points x and y of K, $x\,R\,y$ if x cuts p from y in M or $x=p$. Using Lemma 2, one can prove that R is a natural ordering of K as defined by G. T. Whyburn [5, p. 41]. Hence there exists an arc A not necessarily in S containing K such that p and q are endpoints of A and R is the order induced on K from A [5, Theorem 6.4, p. 56].

Let E be a component of $A-K$ with endpoints a and b. Assume without loss of generality that either a cuts p from b in M or $a=p$. Suppose there exists a point x such that x cuts a from b in M. One can prove that the point x belongs to K, $a\,R\,x$ and $x\,R\,b$. Hence x must belong to E. This contradicts the assumption that E is a subset of $A-K$. Therefore no point cuts a from b in M. Let C denote the set of components of $A-K$. It follows from Jones' cyclic connectivity theorem that for each element E of C, there exists a simple closed curve $J(E)$ in M which contains the endpoints of E. Since M does not separate S, there exists a disk $N(E)$ in M such that the endpoints of E are in $N(E)$. Note that if C is finite, one can easily define an arc in M with endpoints p and q.

Assume that C is infinite. For each element E of C define E^* to be the straight line segment in S which has the endpoints of E as endpoints. Since M is semiaposyndetic, for each positive real number ϵ, the set consisting of all elements E of C such that E^* is not ϵ-spanned by a disk in M is finite. For each positive integer n, let C_n be the finite set consisting of all elements E of C such that either the diameter of E^* is greater than or equal to $1/2n$, or E^* is not $1/2n$-spanned by a disk in M. Let $H_1=C_1$, and, for $n=2, 3, 4, \cdots$, let $H_n=C_n-C_{n-1}$. For each element E of C, define the arc-segment $M(E)$ as follows. Assume that a and b are the endpoints of E. There exists an integer n such that E belongs to H_n. If $n=1$, define $M(E)$ to be an arc-segment in $N(E)$ with endpoints a and b. According to Lemma 1, if $n>1$, there exists an arc-segment $M(E)$ in M with endpoints a and b such that for each point x of $M(E)$, $d(x, E^*)\leqq 1/(n-1)$. One can prove that for each element X of C, $(K\cup\bigcup_{E\in C-\{X\}} M(E))\cap M(X)=\emptyset$. For each element E of C, let f_E be a homeomorphism from E onto $M(E)$. Define the function f from A to $K\cup\bigcup_{E\in C} M(E)$ as follows. For each point x of K, define $f(x)=x$. If x is a point of $A-K$, define $f(x)=f_E(x)$ $(x\in E)$. The function f is a homeomorphism. Hence $K\cup\bigcup_{E\in C} M(E)$ is an arc in M from p to q. It follows that M is arcwise connected.

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