BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 4, July 1971

THE PLEMELJ DISTRIBUTIONAL FORMULAS[1]

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Communicated by Peter Lax, December 9, 1970

1. **Introduction.** Let $(\mathcal{E}')$ be the space of Schwartz distributions with compact supports defined on $\boldsymbol{R}$. Let $(\mathcal{O}'_\alpha)$ be the space of distributions defined on the space $(\mathcal{O}_\alpha)$ of all infinitely differentiable complex-valued functions f on $\boldsymbol{R}$ such that $f(t)=O(|t|^\alpha)$ and $f^{(p)}(t)=O(|t|^\alpha)$ for all p $(|t|\to\infty)$.

In this announcement we extend the famous Plemelj formulas to the distributions in $(\mathcal{E}')$ or $(\mathcal{O}'_\alpha)$. A distributional extension in another direction has been given in [**1**]. The overlap with the present approach is little. The Plemelj numerically-valued relations are discussed in detail in [**2**], [**5**].

Paralleling the classical version, we will consider distributions T that are contained in $(\mathcal{E}')$ or $(\mathcal{O}'_\alpha)$ and define the generalized Cauchy integral of T by $\hat{T}(z)=(1/2\pi i)\ \langle T_t, 1/(t-z)\rangle$, $\operatorname{Im}(z)\neq 0$.

2. **Statement of results.** In what follows, D^+ and D^- denote the open upper and the open lower half-planes, respectively.

THEOREM 1. *If* $T\in(\mathcal{E}')$ *and*

$$\hat{T}^{\pm}(z) = \frac{1}{2\pi i}\left\langle T_t, \frac{1}{t-z}\right\rangle$$

for $z\in D^\pm$, *then* $\hat{T}^\pm=\lim_{\epsilon\to+0}\hat{T}^\pm(t\pm i\epsilon)$ *exist in* $(\mathcal{D}')$ *and*

$$\hat{T}^+ - \hat{T}^- = T, \tag{1}$$

$$\hat{T}^+ + \hat{T}^- = -\frac{1}{\pi i}\left(T * \mathrm{vp}\frac{1}{t}\right). \tag{2}$$

THEOREM 2. *If* $T\in(\mathcal{O}'_\alpha)$ *for* $-1\leqq\alpha<0$ *and*

$$\hat{T}^{\pm}(z) = \frac{1}{2\pi i}\left\langle T_t, \frac{1}{t-z}\right\rangle$$

AMS 1970 *subject classifications*. Primary 30A88, 46F10.

Key words and phrases. Distributions in $(\mathcal{E}')$, $(\mathcal{O}'_\alpha)$, convergence of distributions, decomposition of a distribution, generalized Cauchy integral, Plemelj distributional formulas, distributional analytic continuation, Hilbert boundary problem.

[1] The results (with applications) presented in this note were delivered at the Oberwolfach Conference on Functional Analysis (October, 1970).

for $z \in D^{\pm}$, *then* $\hat{T}^{\pm} = \lim_{\epsilon \to +0} \hat{T}^{\pm}(t \pm i\epsilon)$ *exist in* $(\mathcal{O}'_\alpha)$ *and the formulas* (1) *and* (2) *are valid.*

If $T \in (\mathcal{O}'_\alpha)$ *for* $\alpha \geqq -1$ *the same formulas are valid in* $(\mathcal{D}')$.

THEOREM 3. *If* $T \in (\mathcal{O}'_\alpha)$ *for arbitrary fixed* α *and*

$$\hat{T}^{\pm}(z) = \frac{k!}{2\pi i}\left\langle T_t, \frac{1}{(t-z)^{k+1}}\right\rangle$$

for $z \in D^{\pm}$, *then*

$$\hat{T}^{+} - \hat{T}^{-} = T^{(k)},$$

$$\hat{T}^{+} + \hat{T}^{-} = -\frac{1}{\pi i}\left(T^{(k)} * \operatorname{vp}\frac{1}{t}\right)$$

in $(\mathcal{D}')$. *One supposes* $1/(t-z)^{k+1} = O(|t|^{\alpha})$, $\operatorname{Im}(z) \neq 0$.

THEOREM 4. *In the above theorems the decomposition of distributions into the difference of two distributions is unique.*

Proofs are based on the results in [3], [4] involving a criterion for a linear form to be a distribution, theorems on convergence of distributions and certain facts on the distributional analytic continuation. A complete account will appear elsewhere.

The announced theorems are applicable directly on some classical problems formulated in the sense of distributions in $(\mathcal{E}')$ and $(\mathcal{O}'_\alpha)$ (for example, a Hilbert boundary problem for the half-planes, singular convolution equations).

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