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L^p BOUNDEDNESS OF CERTAIN CONVOLUTION OPERATORS

BY E. M. STEIN

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We announce here several results dealing with the boundedness of convolution operators on $L^p(R^n)$. These results make use of the idea that boundedness on H^1 often holds when weak type (1, 1) estimates apply (see [5, Chapter VII]), and the recent discovery of Fefferman [3] characterizing the dual of H^1.

Our first result states in effect that the complex intermediate spaces between $H^1(R^n)$ and $L^2(R^n)$ are the $L^p(R^n)$. We state this precisely in a particular but useful case. Let $z \to T_z$ be a mapping from the closed strip, $0 \leqq R(z) \leqq 1$, to bounded operators on $L^2(R^n)$, which is assumed analytic in the interior and strongly continuous and uniformly bounded in the closed strip.

THEOREM 1. *Suppose*

$$\sup_{-\infty<y<\infty} \|T_{iy}(f)\|_{H^1} \leqq M_0\|f\|_{H^1}, \qquad f \in H^1 \cap L^2, \tag{1}$$

$$\sup_{-\infty<y<\infty} \|T_{1+iy}(f)\|_{L^2} \leqq M_1\|f\|_{L^2}, \qquad f \in L^2. \tag{2}$$

Then $\|T_t(f)\|_p \leqq M_t\|f\|_p$, $f \in L^p \cap L^2$, *if* $0<t<1$, $1/p = 1-t/2$.

This theorem allows one to obtain certain sharp estimates which did not fall under the scope of previous methods. We give two examples.

First, let $K(x)$ be a distribution of compact support, locally integrable away from the origin, and whose Fourier transform $\hat{K}(x)$ is a function. Assume (following Fefferman [2]) that for some θ, $0 \leqq \theta < 1$,

$$|\hat{K}(x)| \leqq A(1+|x|)^{-n\theta/2}, \tag{i}$$

$$\int_{|x| \geqq 2|y|^{1-\theta}} |K(x-y) - K(x)|\, dx \leqq A, \qquad |y| \leqq 1. \tag{ii}$$

THEOREM 2. $|x|^\gamma \hat{K}(x)$ *is a bounded multiplier for* $(L^p(R^n), L^p(R^n))$ *if* $|(1/p) - \frac{1}{2}| = \frac{1}{2} - \gamma/n\theta$, $\gamma > 0$.

An instance of the above arises with $|x|^\gamma \hat{K}(x) = \theta(x)|x|^{-\beta} \exp i|x|^\alpha$,

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where θ vanishes near the origin, is C^∞, and identically one for large x, $0<\alpha<1$, and $\beta=(n\alpha/2)-\gamma$.

THEOREM 3. *Suppose $d\mu$ is a finite Borel measure whose Fourier transform $\hat{\mu}(x)$ is $O(|x|^{-\delta})$, $|x|\to\infty$, $\delta>0$. Then $|x|^\gamma\hat{\mu}(x)$ is a multiplier for $(L^p(R^n), L^p(R^n))$, if $|(1/p)-\frac{1}{2}|=\frac{1}{2}-\gamma/2\delta$.*

As a corollary of the theorem we can take σ to be the uniform mass distributed on the surface of the unit sphere in R^n. Then $((\partial/\partial x)^\alpha\sigma)\hat{}$ is a multiplier for $(L^p(R^n), L^p(R^n))$ if $|(1/p)-\frac{1}{2}|=\frac{1}{2}-|\alpha|/(n-1)\geqq 0$. This corollary leads to certain new estimates for the wave equation, and it applies also to the problem of spectral synthesis for the algebra $A_p(R^n)$. (For the latter, see Eymard [1].)

We state also a further property of H^1 which can be viewed as a sort of converse to Theorem 4 in [3]. Again we state only a special case of the result. Let φ be a C^1 function on R^n of compact support and set $\varphi_\epsilon(x)=\epsilon^{-n}\varphi(x/\epsilon)$. Let $f^*(x)=\sup_{\epsilon>0,|t|\leq\epsilon}|(f*\varphi_\epsilon)(x-t)|$.

THEOREM 4. *If $f\in H^1$ then $f^*\in L^1$ and*

$$\|f^*\|_{L^1}\leqq A\|f\|_{H^1}.$$

Detailed proofs and extensions of these results to H^p, $p<1$, will appear in [4].

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PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540