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MULTIPLICATIVE OPERATOR FUNCTIONALS OF A MARKOV PROCESS

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1. **Introduction.** Let $X=(x(t), \zeta, \mathfrak{M}_t, P_x)$ be a right continuous Markov process on a state space $(E, \mathcal{B})$.[2] Let L be a fixed Banach space. A *multiplicative operator functional* (MOF) of (X, L) is a mapping $(t, \omega)\to M(t, \omega)$ of $[0, \infty)\times\Omega$ to bounded operators on L which possesses the following properties:

(1a) $\omega\to M(t, \omega)f$ is $\mathfrak{M}_t$ measurable for each $t\geqq 0, f\in L$.

(1b) $t\to M(t, \omega)f$ is right continuous a.s. for each $f\in L$.

(1c) $M(t+s, \omega)f=M(t, \omega)M(s, \theta_t\omega)f$ a.s. for each $s, t\geqq 0, f\in L$.

(1d) $M(0, \omega)$ is the identity operator on L.

If M is a multiplicative operator functional of (X, L) the *expectation semigroup* is defined on the direct sum Banach space $\tilde{L}=\bigoplus_E L$ by the equation

$$(\tilde{T}(t)\tilde{f})_x = E_x[M(t, \omega)f_{x(t,w)}]. \tag{1.1}$$

The MOF concept has appeared in several places recently. We were led to the idea by the work of Griego and Hersh [4], [5] who constructed examples of an MOF when X is a Markov chain with a finite number of states and L is arbitrary. Here $M(t, \omega)$ is a finite random product of semigroups. Earlier Babbitt [1] had studied the case X=Wiener process on R^n, L=finite-dimensional vector space. In this case $M(t)$ is a solution to a system of Itô stochastic differential equations. If X is a Poisson process and L is a Banach space of continuous functions on R^1, we can specialize the MOF concept to represent the semigroups studied by Çinlar and Pinsky in a problem in storage theory. In this case the infinitesimal operator of the associated semigroup is an integro-differential operator. Further applications will be discussed in another publication.

2. **Main results.** Here we will give the notations and state the main results. Proofs will not be given. Detailed proofs will appear elsewhere.

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[2] See Dynkin [3] for the definition and notations for Markov processes.

If L is a Banach space and E is an arbitrary index set, $\tilde{L}$ is the Banach space of all tuples $\tilde{f}=(f_x)_{x\in E}$ where $f_x \in L$ for each x; $\tilde{L}$ is equipped with componentwise addition and scalar multiplication; we use the norm $\|\tilde{f}\|_{\tilde{L}}=\sup_x\|f_x\|$. A *contractive* MOF is an MOF with the additional property that $\|M(t, \omega)f\| \leqq \|f\|$ a.s. for $t\geqq 0, f\in L$.

If X has no instantaneous states, let $0=\tau_0<\tau_1<\tau_2< \cdots$ be the jump times of $x(t)$ and let $N(t)$ be the number of jumps in $[0, t]$.

THEOREM 1. *Let $E=\{1, 2, \cdots, N\}$ and let X be a finite Markov chain on E with $\zeta=+\infty$. Let M be a contractive multiplicative operator functional of (X, L). Then there exist strongly continuous contraction semigroups $T_1(t), \cdots, T_N(t)$ on L and bounded operators $\{\Pi_{\alpha\beta}\}_{1\leq\alpha\neq\beta\leq N}$ on L such that for $t\geqq 0, f\in L$,*

$$(2.1)\quad M(t)f = \prod_{i=1}^{N(t)} T_{x(\tau_{i-1})}(\tau_i - \tau_{i-1})\, \Pi_{x(\tau_{i-1})x(\tau_i)}\, T_{x(\tau_{N(t)})}(t-\tau_{N(t)})f.$$

The key step in the proof of this result is the observation that $M(t, \omega)$ is a semigroup up until the first jump time of X. More precisely we have a

LEMMA. *If $0\leqq s, t$ and $t+s<\tau_1(\omega)$, then*

$$M(t+s, \omega)=M(t, \omega)M(s, \omega) \quad a.s.$$

In the case of a continuous MOF we can specialize the form (2.1) and also relax the hypothesis.

THEOREM 2. *Let $E=\{1, 2, \cdots, N\}$ and let X be a finite Markov chain on E with $\zeta=+\infty$. Let M be a multiplicative operator functional of (X, L) such that the mapping $t\to M(t, \omega)f$ is continuous from $[0, \infty)$ to L for each $f\in L$, $\omega\in\Omega$. Then there exist strongly continuous semigroups $T_1(t), \cdots, T_N(t)$* on *L such that, for $f\in L$,*

$$(2.2)\quad M(t)f = T_{x(0)}(\tau_1)T_{x(\tau_1)}(\tau_2-\tau_1) \cdots T_{x(\tau_{N(t)})}(t-\tau_{N(t)})f.$$

REMARK. The form (2.2) is termed a *random evolution* by Griego and Hersh. In some sense $M(t)$ evolves by the random equation $dM/dt=M(t)A_{x(t)}$ where A_x is the infinitesimal operator of the semigroup $T_x(t)$.

3. Examples and other applications. We limit this discussion to two classes of examples of MOF's. Other applications will be discussed elsewhere.

I. Let $X=(x(t), +\infty, \mathfrak{M}_t, P_x)$ be a separable Wiener process on R^n. Let $B_1, \cdots, B_n$ and V be bounded continuous complex $N\times N$

matrix-valued functions on R^n and let $\tilde{D}(B;V)$ be the closure in $L^2(R^n;\, C^N)$ of the differential operator

$$D(B;V) = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2 + \sum_{j=1}^{n} B_j(x)\frac{\partial}{\partial x_j} + V(x)$$

with domain $C_0^\infty(R^n;\, C^N)$. Babbitt [1] has shown that there exists a complex $N\times N$ matrix-valued function $\alpha(t,\omega)=\alpha$ on $\Omega\times[0,\infty)$ such that α is adapted to $\mathfrak{M}_t$, the map $t\to\alpha(t,\omega)$ is continuous on $[0,\infty)$ a.s., and α is a solution of the stochastic integral equation

$$\alpha(t,\omega) = I + \sum_{j=1}^{n}\int_0^t \alpha(\tau,\omega)B_j(x(\tau))\,dx_j(\tau) + \int_0^t \alpha(\tau,\omega)V(x(\tau))\,d\tau$$

a.s. with respect to P_x; here I is the identity matrix and

$$\sum_{j=1}^{n}\int_0^t \alpha(\tau,\omega)B_j(x(\tau))\,dx_j(\tau)$$

is an Itô stochastic integral. Furthermore α satisfies the bound $E_x\{|\alpha(t)|^2\}\leqq Q_T$ for some constant Q_T independent of x, for $0\leqq t\leqq T$.

Now we note that $\alpha(t+\cdot,\omega)$ and $\alpha(t,\omega)\alpha(\cdot,\theta_t\omega)$ (juxtaposition means matrix multiplication) both solve the same stochastic integral equation. Thus by the uniqueness theorem for stochastic integral equations (see [6, p. 54 ff.]) they must be equal a.e. Thus $\alpha(t,\omega)$ satisfies the multiplicative law $\alpha(t+s,\omega)=\alpha(t,\omega)\alpha(s,\theta_t\omega)$. The formula

$$T_t\phi(x) = E_x[\alpha(t,\omega)\phi(x(t))]$$

according to Babbitt defines a semigroup on $L^2(R^n;\, C)$ whose infinitesimal operator is $\tilde{D}(B;V)$. Conversely, if $\alpha(t,\omega)$ is a matrix which satisfies the above multiplicative law together with certain smoothness conditions, then α satisfies a stochastic integral equation of the above type, as we shall show in another publication.

II. As a second example, let X be a Poisson process with rate a $(0<a<\infty)$ with jump times $0<\tau_1<\tau_2<\cdots<\infty$. Let L be the Banach space of all continuous functions on $[0,\infty)$ which vanish at ∞. Let $T(t)$ be the group on L with $T(t)f(x)=f(k_t(x))$ where $k_t(x)$ is a solution of the o.d.e $\dot{k}_t=-r(k_t)$, $k_0(x)=x$. Here r is a strictly increasing differentiable function with $r(0)=0$. Let Π be the convolution operator on L given by $\Pi f(x)=\int_0^\infty f(x+y)\nu(dy)$, where ν is a fixed probability measure on $[0,\infty)$. Then the formula

$$M(t) = T(\tau_1)\Pi T(\tau_2-\tau_1)\Pi\cdots\Pi T(t-\tau_{N(t)})$$

defines a multiplicative operator functional on (X, L) with the property that the expectation semigroup leaves invariant the class $\{\tilde{f}=(f, f, \cdots), f\in L\}$. A calculation shows that the infinitesimal operator of the expectation semigroup is given by an extension of the operator

$$Af = -r(x)\frac{\partial f}{\partial x} + a\int_0^\infty (f(x+y) - f(x))\nu(dy)$$

acting on smooth functions. This semigroup and its limit as $t\rightarrow\infty$ were considered in detail in [2].

The author recently became aware of the work of D. W. Stroock (Comm. Pure Appl. Math. **23** (1970), 447–457). This paper extends the work of Babbitt to include, in particular, coefficients (B_j, V) which depend on t and are merely bounded and measurable.

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