ONE-PARAMETER SEMIGROUPS OF ISOMETRIES

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Let $t \to V_t$ for $t \geqq 0$ be a strongly continuous one-parameter semigroup of isometries on a Hilbert space H. The easiest example of such a semigroup which is not unitary is given by considering the Hilbert space $\tilde{H} = L^2(0, \infty)$ consisting of those Lebesgue square-integrable functions on $(-\infty, \infty)$ which are supported on $(0, \infty)$. On $\tilde{H}$, we consider the (nonunitary) isometries

$$(T_t f)(x) = f(x - t).$$

Recently, the C^*-algebra $\mathfrak{A}(T_t : t \geqq 0)$ generated by the semigroup $t \to T_t$ has been studied in detail [**2**], [**3**], [**4**].

In this note, we show that for any strongly continuous one-parameter semigroup of isometries $t \to V_t$ with V_{t_0} not unitary for some t_0, $\mathfrak{A}(V_t : t \geqq 0)$ is *-isomorphic with $\mathfrak{A}(T_t : t \geqq 0)$. The proof is modelled after the corresponding result for C^*-algebras generated by a single isometry [**1**].

The main fact that we use is a generalization due to Cooper [**6**, p. 142] of the Wold decomposition of a single isometry [**5**, p. 109]. This generalization states that for $t \to V_t$, $t \geqq 0$, a strongly continuous one-parameter semigroup of isometries on H, there is a Hilbert space K with a strongly continuous one-parameter unitary semigroup $t \to U_t$ on K, there is a cardinal α, and there is an isometry U from H onto $K \oplus \tilde{H} \oplus \cdots \oplus \tilde{H} \oplus \cdots$ where $\tilde{H}$ occurs with multiplicity α, such that

$$UV_tU^* = U_t \oplus T_t \oplus \cdots \oplus T_t \oplus \cdots.$$

The multiplicity α is a unitary invariant which can be read off from the infinitesimal generator of $t \to V_t$ [**6**, p. 142].

In case $K = \{0\}$, we say that $t \to V_t$ is *purely nonunitary* [**6**, p. 136]. For such semigroups, the multiplicity α is the *only* unitary invariant. A very general way of generating such semigroups is to consider for any measure $d\mu$ which is positive, of bounded variation, and singular with respect to Lebesgue measure on the unit circle T, the singular inner functions [**5**, p. 66] $\phi_t^\mu(e^{i\theta})$ which are the boundary values of

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$$\exp\left\{-t\int \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\,d\mu(\alpha)\right\}, \qquad |z| < 1.$$

It is then easy to check that for f in the usual Hardy space $H^2(T)$ [5, p. 39],

$$(M_t^\mu f)(e^{i\theta}) = \phi_t^\mu(e^{i\theta})f(e^{i\theta})$$

defines a strongly continuous one-parameter semigroup of isometries for $t \geqq 0$. The second result of this note shows that $t \rightarrow M_t^\mu$ is purely nonunitary and characterizes the multiplicity $\alpha(\mu)$ of $t \rightarrow M_t^\mu$ directly in terms of the measure μ.

THEOREM A. *Let $t \rightarrow V_t$, $t \geqq 0$, be a strongly continuous one-parameter semigroup of isometries with V_{t_0} nonunitary for some t_0. Then the C^*-algebra $\mathfrak{A}(V_t : t \geqq 0)$ generated by the V_t is *-isomorphic with $\mathfrak{A}(T_t : t \geqq 0)$.*

PROOF. Applying the decomposition of Cooper to $t \rightarrow V_t$, we see that the problem is reduced to studying

$$\mathfrak{A} = \mathfrak{A}(U_t \oplus T_t \oplus \cdots \oplus T_t \oplus \cdots : t \geqq 0),$$

where T_t occurs with multiplicity $\alpha \geqq 1$. Now $\mathfrak{A}$ is just the norm-closure of direct sums of the form

$$\sum_{j=1}^{n} a_{t_j,s_j} U_{t_j} U_{s_j}^* \oplus \sum_{j=1}^{n} a_{t_j,s_j} T_{t_j} T_{s_j}^* \oplus \cdots.$$

The mapping Φ which sends such a direct sum to

$$\sum_{j=1}^{n} a_{t_j,s_j} T_{t_j} T_{s_j}^*$$

clearly extends to a *-homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}(T_t : t \geqq 0)$. To check that Φ is actually a *-isomorphism, it suffices to show that

$$\left\| \sum_{j=1}^{n} a_{t_j,s_j} U_{t_j} U_{s_j}^* \right\| \leqq \left\| \sum_{j=1}^{n} a_{t_j,s_j} T_{t_j} T_{s_j}^* \right\|.$$

The structure of the algebra $\mathfrak{A}(T_t : t \geqq 0)$ has been described in [2], [3]. We use the fact that $\mathfrak{A}(T_t : t \geqq 0)$ contains a proper closed two-sided ideal $\mathfrak{C}$ (the commutator ideal) and for R the real line,

$$\inf_{C \in \mathfrak{C}} \left\| \sum_{j=1}^{n} a_{t_j,s_j} T_{t_j} T_{s_j}^* + C \right\| = \sup_{x \in R} \left| \sum_{j=1}^{n} a_{t_j,s_j} \exp[i(t_j - s_j)x] \right|.$$

It follows that it will be enough to show that

$$\left\| \sum_{j=1}^{n} a_{t_j,s_j} U_{t_j} U_{s_j}^* \right\| \leqq \sup_{x \in R} \left| \sum_{j=1}^{n} a_{t_j,s_j} \exp[i(t_j - s_j)x] \right|.$$

Now noting that $t \to U_t$ is a strongly continuous semigroup for $t \geqq 0$, we see that U_t commutes with U_s^* and $t \to U_t$ can be extended to a unitary representation of R by defining $U_{-t} = U_t^*$ for $t \geqq 0$. The desired inequality is obtained by observing that for some self-adjoint (not necessarily bounded) A on H,

$$\langle U_t f, g \rangle = \int_{x \in \sigma(A)} e^{itx} d\langle E(x)f, g\rangle, \qquad t \in (-\infty, \infty),$$

where f and g are in H, A is the infinitesimal generator for $t \to U_t$, $E(x)$ is the spectral family for A, and $\sigma(A)$ is the spectrum of A ($\sigma(A) \subset R$) [6, p. 134]. Hence, using the fact that for $\|f\| = 1$

$$\int_{x \in \sigma(A)} d\langle E(x)f, f\rangle = 1,$$

we see that for $\|f\| = 1$

$$\begin{aligned} \left\| \sum_{j=1}^{n} a_{t_j,s_j} U_{t_j} U_{s_j}^* f \right\|^2 &= \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{a}_{t_j,s_j} a_{t_k,s_k} \langle U_{s_j - t_j + t_k - s_k} f, f\rangle \\ &= \int_{x \in \sigma(A)} \left| \sum_{j=1}^{n} a_{t_j,s_j} \exp[i(t_j - s_j)x] \right|^2 d\langle E(x)f, f\rangle \end{aligned}$$

and the desired inequality follows.

THEOREM B. *The strongly continuous one-parameter semigroup of isometries $t \to M_t^\mu$ described above is purely nonunitary and the multiplicity $\alpha(\mu)$ is determined as follows: $\alpha(\mu) = n$ if the support of μ consists of exactly n points, $\alpha(\mu) = \infty$ otherwise.*

PROOF. Let us first show that if w is any nonconstant inner function [5, p. 62], then for f in $H^2(T)$, the isometry $(M_w f)(z) = w(z)f(z)$ is purely nonunitary. Otherwise, for some f in $H^2(T)$ with $\|f\| = 1$, $\|M_w^{*n} f\| = 1$ for $n = 1, 2, \cdots$, or equivalently, for g_n in $H^2(T)$

$$(*) \qquad f = w^n g_n, \qquad n = 1, 2, \cdots.$$

Thus, if $w(z_0) = 0$ for $|z_0| < 1$ then f has a zero of infinite order at z_0, which is impossible. Thus, w is purely singular and nonconstant so

$$w(z) = \exp\left\{ -\int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\nu(\alpha) \right\}$$

where ν is a uniquely determined finite positive singular measure on

T, $0<\nu(T)<\infty$ [5, p. 66]. Equating the singular parts of the functions in (*), we see that for

$$f_{\text{sing}}(z) = \exp\left\{-\int \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\,d\sigma(\alpha)\right\},$$

$$(g_n)_{\text{sing}}(z) = \exp\left\{-\int \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\,d\tau_n(\alpha)\right\}$$

where σ and τ_n are finite positive singular measures on T, we have $\sigma=n\nu+\tau_n$ so $\sigma(T)\geqq n\nu(T)$ for $n=1, 2, \cdots$. Since $\sigma(T)<\infty$ and $\nu(T)>0$, we have a contradiction.

We remark further that the defect of M_w (the dimension of kernel(M^*)) is finite if and only if w is a finite Blaschke product, and that in this case the defect equals the number of terms in the Blaschke product. We now prove this assertion. Certainly, if $w=\prod_{k=1}^N w_k$, where the w_k are nonconstant inner functions, then $M_w=\prod_{k=1}^N M_{w_k}$. Each of the M_{w_k} are purely nonunitary isometries and so have defect at least one. Thus, since

$$\text{defect}(M_w) = \sum_{k=1}^{N} \text{defect}(M_{w_k})$$

(this follows from elementary index-type argument), we have defect $(M_w)\geqq N$. It follows easily that if w has a nonconstant singular part or a Blaschke part with infinitely many zeros, defect$(M_w)=\infty$. If

$$w(z) = \lambda \prod_{k=1}^{N}\left(\frac{z-a_k}{1-\bar{a}_k z}\right)$$

where λ is a constant, $|\lambda|=1$ and $|a_k|<1$, then

$$\text{defect}(M_w) = \sum_{k=1}^{N} \text{defect}(M_{((z-a_k)/(1-\bar{a}_k z))})$$

and each $M_{((z-a_k)/(1-\bar{a}_k z))}$ has defect one.

Now by the result of Foiaş and Nagy [6, p. 142], $\alpha(\mu)$ is just the defect of the isometry obtained by taking the Cayley transform of the infinitesimal generator of the semigroup $t\to M_t^\mu$. The infinitesimal generator of the semigroup is M_ψ, where ψ is the function

$$\psi(z) = -\int \frac{e^{i\alpha}+z}{e^{i\alpha}-z}\,d\mu(\alpha), \qquad |z|<1.$$

The Cayley transform of M_ψ is M_w, where $w=(1+\psi)/(1-\psi)$. Since $\text{Re}\,\psi(z)<0$, we see that $|w(z)|<1$ for $|z|<1$. Since μ is singular,

$$\lim_{\rho \to 1^-} \operatorname{Re} \psi(\rho e^{i\theta}) = 0 \quad \text{a.e. } (\theta)$$

so w is an inner function.

By the foregoing, it suffices to show that w is a finite Blaschke product of n terms if and only if support(μ) is a finite set with n points. But if support(μ) is finite (with n points) then $w(z)$ is a rational function. The only rational inner functions are finite Blaschke products [5, p. 76] so w has the form

$$w(z) = \lambda \prod_{k=1}^{m} \left(\frac{z - a_k}{1 - \bar{a}_k z} \right)$$

where $|\lambda| = 1$ and $|a_k| < 1$. But the zeros of $w(z)$ in the plane are those of

$$1 + \psi(z) = 1 - \sum_{k=1}^{n} \frac{\exp[i\alpha_k] + z}{\exp[i\alpha_k] - z} t_k, \qquad t_k > 0,$$

and multiplying by $\prod_{k=1}^{n}(\exp[i\alpha_k] - z)$ we see that $1 + \psi(z)$ has exactly n zeros so that $n = m$. Conversely, suppose

$$w = \lambda \prod_{k=1}^{n} \left(\frac{z - a_k}{1 - \bar{a}_k z} \right), \qquad |\lambda| = 1.$$

Then

$$-\int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) = \psi(z) = \frac{w(z) - 1}{1 + w(z)}$$

is a rational function and so has at most finitely many points of T in its natural boundary. But then support(μ) contains only those points [5, p. 68], and so is finite. This completes the proof.

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