HOMOGENEOUS POSITIVELY PINCHED RIEMANNIAN MANIFOLDS

BY NOLAN R. WALLACH

Communicated by M. H. Protter, January 26, 1970

1. **Introduction.** The purpose of this paper is to announce several results on homogeneous positively pinched manifolds. We give, modulo two examples, a complete classification of even dimensional homogeneous spaces that admit positively pinched homogeneous Riemannian structures (see §5, Theorem 5.2). These spaces are all diffeomorphic with rank one symmetric space except for two possible exceptions $SU(3)/T$ (T the maximal torus in $SU(3)$) and

$$Sp(3)/SU(2) \times SU(2) \times SU(2).$$

2. **Preliminary results.** Let G be a compact connected Lie group and let K be a closed subgroup of G. Let $M=G/K$ and suppose that M carries a G-invariant positively pinched Riemannian structure (that is all sectional curvatures are bounded below by a positive constant).

PROPOSITION 2.1. *If M is even dimensional then G is semisimple. If M is odd dimensional then G is either semisimple or G has a one-dimensional center.*

This result is proved by analyzing directly the curvature of M as a 4-linear form on a suitably chosen $\mathrm{Ad}(K)$-invariant complement $\mathfrak{p}$ to the Lie algebra of K, in the Lie algebra of G, $\mathfrak{g}$ (in particular $\mathfrak{p}$ is chosen so that it contains the center of $\mathfrak{g}$). The even dimensional statement will be considerably strengthened in the next section.

The next result indicates the difficulties in using a Lie theoretic, algebraic approach to the general study of homogeneous positively pinched Riemannian manifolds.

PROPOSITION 2.2. *Let (G, K) be as in Proposition 1. Let T be a maximal torus in K. Let $C(T)$ be the centralizer of T in G. Let $p_0=eK$ (the identity coset in M) and let $M_T=C(T)p_0$. Then*

(1) *M_T is totally geodesic in M.*

(2) *$C(T)/T$ is either one dimensional or semisimple.*

AMS Subject Classifications. Primary 5372, 5366.

Key Words and Phrases. Homogeneous, Riemannian structure, Killing field, Lie group, isometry.

We propose

CONJECTURE. Let G be a compact, connected, simply connected Lie group. Suppose that G has a left invariant positively pinched Riemannian structure. Then $G = \mathrm{SU}(2)$.

If the conjecture were true then Proposition 2.1 and Proposition 2.2 would imply

COROLLARY TO CONJECTURE. *Let G be a compact connected Lie group and let K be a closed subgroup of G such that G acts faithfully on $M = G/K$ and such that M admits a G-invariant positively pinched Riemannian structure.*

(i) *If M is even dimensional then the rank of K is equal to the rank of G.*

(ii) *If M is odd dimensional then the rank of G = rank of $K+1$.*

The above corollary to the conjecture is in fact equivalent to it. We will see in §3 that assertion (i) of the corollary is true and a direct consequence of a result of Berger [2] (see also Weinstein [6]).

3. Implications of a result of Berger.

PROPOSITION 3.1. *Let M be an even dimensional, connected, orientable, positively pinched Riemannian manifold. Let G be a compact connected Lie group acting by isometries effectively and transitively on M. Let K be the isotropy group, in G, of a point in M. Then G is a simple center free Lie group and G and K have the same rank.*

This result follows directly from a result of Berger [2] which says that every Killing field on an even dimensional positively pinched manifold must have a zero. See also Weinstein [2] for a strengthening of this result.

Using Proposition 3.1 and a result of O'Neil [4], we have

COROLLARY 3.1. *Let G be a compact simply connected Lie group admitting a positively pinched left invariant Riemannian structure then G is odd dimensional and simple.*

4. Main lemma.

Let G, K be as in the beginning of §2. Let $M = G/K$ and assume that M is even dimensional. Then, by Proposition 3.1, G has a maximal torus, T, contained in K. Let Δ be the root system (written additively) of G relative to T and let Δ_1 be the root system of K relative to T. Then $\Delta_1 \subset \Delta$. Let $\Delta - \Delta_1$ denote the complement to Δ_1 in Δ.

LEMMA 4.1. *If α, $\beta \in \Delta - \Delta_1$ and $\alpha \neq \pm\beta$ then $\alpha+\beta$ or $\alpha-\beta \in \Delta$.*

5. The classification theorem. Let G be a compact, connected, simply connected, simple, Lie group and let K be a closed subgroup of G such that G and K share a maximal torus T. Let Δ be the root system of G relative to T and let Δ_1 be the root system of K relative to T. Then (G, K) is said to satisfy condition (A) if

(A) $\alpha, \beta \in \Delta - \Delta_1$, $\alpha \neq \pm\beta$ implies $\alpha+\beta$ or $\alpha-\beta \in \Delta$.

THEOREM 5.1. *The only pairs* (G, K) *as above satisfying condition* (A) *are*

1. $(\mathrm{SU}(n+1), \mathrm{U}(n))$ *and* $G/K = CP^n$ *complex n dimensional projective space.*
2. $(\mathrm{SU}(3), T)$ *where* T *is a maximal torus of* $\mathrm{SU}(3)$.
3. $(\mathrm{Spin}(2n+1), \mathrm{Spin}(2n))$ *and* $G/K = S^n$ *the n sphere.*
4. $(\mathrm{Sp}(n), \mathrm{Sp}(n-1) \times \mathrm{SU}(2))$ *and* G/K *is n dimensional quaternionic projective space.*
5. $(\mathrm{Sp}(n), \mathrm{Sp}(n-1) \times T')$ *where* T' *is a circle group and* G/K *is diffeomorphic with* CP^{2n-1}.
6. $(\mathrm{Sp}(3), \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))$.
7. $(F_4, \mathrm{Spin}(9))$ *and* G/K *is the Cayley plane.*
8. $(G_2, \mathrm{SU}(3))$ *and* G/K *is diffeomorphic with* S^6.

The proof of this result is given by a case by case check of the compact simply connected simple Lie groups, using the results of [**4**] and the observation that if (G, K) satisfies condition (A) and if $G \supset K_1 \supset K$, K_1 a closed subgroup of G then (G, K_1) satisfies condition (A).

Combining Proposition 3.1, Lemma 4.1 and Theorem 5.1, we have

THEOREM 5.2. *Let M be a connected, simply connected, even dimensional, positively pinched Riemannian manifold. Let G be a connected, closed transitive subgroup of the isometry group of M. Let $\tilde{G}$ be the universal covering group of G and let $\tilde{K}$ be the isotropy group of a point in M relative to the induced action of $\tilde{G}$ on M. Then $(\tilde{G}, \tilde{K})$ is one of the pairs* 1–8 *of Theorem* 5.1.

Note. Except for 2, 6 the pairs above in Theorem 5.1 are exactly the pairs that Berger [**1**] found in the even dimensional naturally reductive (that is, the Riemannian structure comes from a bi-invariant metric on G) case.

COROLLARY 5.1. *Let M be as in Theorem* 5.2. *Then M is either diffeomorphic to a rank* 1 *symmetric space or M may be one of* $\mathrm{SU}(3)/T$, $\mathrm{Sp}(3)/\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$.

We are unable at this point to prove (or disprove) that the above

exceptional examples have homogeneous positively pinched metrics.
Using Proposition 2.1 and Theorem 5.2 we have

COROLLARY 5.2. *Let G, K be as in Proposition* 2.1. *If G is not semisimple then $G = \mathrm{U}(n)$ or $G = T' \times \mathrm{Sp}(n)$.*

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RUTGERS UNIVERSITY, NEW BRUNSWICK NEW JERSEY 08903