THE SINGULAR SETS OF AREA MINIMIZING RECTIFIABLE CURRENTS WITH CODIMENSION ONE AND OF AREA MINIMIZING FLAT CHAINS MODULO TWO WITH ARBITRARY CODIMENSION[1]

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1. When describing the interior structure of an area minimizing m dimensional locally rectifiable current T in R^{m+1}, one calls a point $x \in \operatorname{spt} T \sim \operatorname{spt} \partial T$ regular or singular according to whether or not x has a neighborhood V such that $V \cap \operatorname{spt} T$ is a smooth m dimensional submanifold of R^{m+1}. As a result of the efforts of many geometers it is known that there exist no singular points in case $m \leqq 6$; a detailed exposition of this theory may be found in [3, Chapter 5]. Recently it was proved in [2] that

$$Z = \partial(E^8 \llcorner R^8 \cap \{x : x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\})$$

is a 7 dimensional area minimizing current in R^8 with the singular point 0. This implies that, for $m > 7$, $E^{m-7} \times Z$ is an m dimensional area minimizing current in $R^{m-7} \times R^8 \simeq R^{m+1}$ with the $m-7$ dimensional singular set $R^{m-7} \times \{0\}$. Here we will show (Theorem 1) that the Hausdorff dimension of the singular set of an m dimensional area minimizing rectifiable current in R^{m+1} never exceeds $m-7$.

Our method also yields the result (Theorem 2) that the Hausdorff dimension of the singular set of an m dimensional area minimizing flat chain modulo 2 in R^{m+p} never exceeds $m-2$, for arbitrary codimension p.

2. We use the terminology of [3]. Given any positive integer m we choose Υ according to [3, 5.4.7] with $n = m+1$ and let

$$\omega(T) = \{x : \Theta^m(\|T\|, x) \geqq \Upsilon\} \quad \text{for } T \in \mathcal{R}_m^{\mathrm{loc}}(R^{m+1}).$$

Whenever $0 \leqq k \in R$ and $A \subset R^{m+1}$ we define $\phi_\infty^k(A)$ as the infimum of the set of numbers $\sum_{B \in G} \alpha(k) 2^{-k} (\operatorname{diam} B)^k$ corresponding to all countable open coverings G of A. We see from [3, 2.10.2] that

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$$\phi_\infty^k(A) = 0 \quad \text{if and only if } \mathcal{H}^k(A) = 0,$$

and from [3, 2.10.19(2)] that

$$\Theta^{*k}(\phi_\infty^k \llcorner A, x) \geqq 2^{-k} \quad \text{for } \mathcal{H}^k \text{ almost all } x \text{ in } A.$$

LEMMA 1. *If $Q_i \in \mathcal{R}_m^{\mathrm{loc}}(R^{m+1})$ and Q_i is absolutely area minimizing with respect to R^{m+1} for each positive integer i,*

$$Q_i \to Q \quad in\ \mathcal{F}_m^{\mathrm{loc}}(R^{m+1}) \quad as\ i \to \infty,$$

and K is a compact subset of $R^{m+1} \sim \mathrm{Clos} \bigcup_{i=1}^\infty \mathrm{spt}\, \partial Q_i$, then

$$\phi_\infty^k[\omega(Q) \cap K] \geqq \limsup_{i\to\infty} \phi_\infty^k[\omega(Q_i) \cap K].$$

PROOF. We observe that if V is any open set containing $\omega(Q) \cap K$, then V contains $\omega(Q_i) \cap K$ for all sufficiently large integers i. Otherwise we could choose a subsequence of points $x_i \in \omega(Q_i) \cap K \sim V$ converging to point $x \in K \sim V$. Since

$$d = \mathrm{dist}\left(K, \bigcup_{i=1}^\infty \mathrm{spt}\, \partial Q_i\right) > 0,$$

we would find whenever $d > r > s > 0$ that $s^{-m}\|Q_i\| U(x_i, s) \geqq \alpha(m)\Upsilon$ according to [3, 5.4.3(3)], with $B(x_i, s) \subset U(x, r)$ for large i, hence

$$\|Q\| U(x, r) \geqq \limsup_{i\to\infty} \|Q_i\| U(x_i, s) \geqq s^m \alpha(m)\Upsilon$$

by [3, 5.4.2]. Thus $\|Q\| U(x, r) \geqq r^m \alpha(m)\Upsilon$ for $0 < r < \delta$, and we could infer that $x \in \omega(Q) \cap (K \sim V) = \emptyset$.

LEMMA 2. *If $T \in \mathcal{R}_m^{\mathrm{loc}}(R^{m+1})$, T is absolutely area minimizing with respect to R^{m+1}, $a \in \mathrm{spt}\, T \sim \mathrm{spt}\, \partial T$ and $\Theta^{*k}[\phi_\infty^k \llcorner \omega(T), a] > 0$, then there exists an oriented tangent cone Q of T at a such that $\mathcal{H}^k[\omega(Q)] > 0$.*

PROOF. Assuming $\Theta^{*k}[\phi_\infty^k \llcorner \omega(T), a] > 2^k c > 0$ and recalling the proof of [3, 5.4.3], in particular the argument on pages 624 and 625, we choose ρ_i and β_i for each positive integer i so that

$$0 < 2\rho_i < i^{-1}\sigma_i, \qquad \phi_\infty^k[\omega(T) \cap B(a, \rho_i)] > \alpha(k)\rho_i^k 2^k c,$$

$$\beta_i^{-1} \in G_i, \qquad \rho_i \leqq (1 - i2^{-i})2\rho_i < \beta_i^{-1} < 2\rho_i.$$

Then $\phi_\infty^k[\omega(T) \cap B(a, \beta_i^{-1})] > \alpha(k)\beta_i^{-k} c$ and the corresponding currents $Q_i = (\mu_{\beta_i} \circ \tau_{-a})_\# T$ satisfy the condition $\phi_\infty^k[\omega(Q_i) \cap B(0, 1)] > \alpha(k)c$. A subsequence of $Q_1, Q_2, Q_3, \cdots$ converges in $\mathcal{F}_m^{\mathrm{loc}}(R^{m+1})$ to an oriented tangent cone Q of T at a, for which $\phi_\infty^k[\omega(Q)\cap B(0,1)]\geqq \alpha(k)c$ according to Lemma 1.

THEOREM 1. *If $T\in \mathcal{R}_m^{\text{loc}}(R^{m+1})$, $m\geqq 7$ and T is absolutely area minimizing with respect to R^{m+1}, then there exists an open set V such that $V\cap \operatorname{spt} T$ is an m dimensional submanifold of class ∞ of R^{m+1} and*

$$\mathcal{H}^k[R^{m+1} \sim (V \cup \operatorname{spt} \partial T)] = 0 \quad \textit{whenever } m - 7 < k \in R.$$

PROOF. We use induction with respect to m. First we will prove the following statement:

If M is an $\mathcal{L}^{m+1}$ measurable set, U is an open subset of R^{m+1},

$$S = [\partial(E^{m+1} \llcorner M)] \llcorner U \in \mathcal{R}_m(R^{m+1})$$

and S is absolutely area minimizing with respect to R^{m+1}, then there exist an open set W such that $W\cap \operatorname{spt} S$ is an m dimensional submanifold of class ∞ of R^{m+1} and

$$\mathcal{H}^k(U \sim W) = 0 \quad \textit{whenever } m - 7 < k \in R.$$

In view of [3, 5.4.7] it suffices to show that

$$\mathcal{H}^k[U \cap \omega(S)] = 0 \quad \text{whenever } m - 7 < k \in R.$$

Assuming the contrary we choose $k>m-7$ and $a\in U\cap\omega(S)$ so that $\Theta^{*k}[\phi_\infty^k \llcorner \omega(S), a]>0$, apply Lemma 2 to obtain an oriented tangent cone C of S at a with $\mathcal{H}^k[\omega(C)]>0$, and infer from [3, 5.4.3(5), (8)] that C is absolutely area minimizing with respect to R^{m+1} and $C=\partial(E^{m+1}\llcorner N)$ for some $\mathcal{L}^{m+1}$ measurable set N. Since $\mathcal{H}^k\{0\}=0$ we can choose $b\in\omega(C)\sim\{0\}$ so that $\Theta^{*k}[\phi_\infty^k\llcorner\omega(C),b]>0$, and repeat the procedure to construct an oriented tangent cone D of C at b such that $\mathcal{H}^k[\omega(D)]>0$, D is absolutely area minimizing with respect to R^{m+1} and $D=\partial(E^{m+1}\llcorner P)$ for some $\mathcal{L}^{m+1}$ measurable set P. We infer from [3, 4.3.16] that D is a cylinder with direction $b/|b|$, from [3, 4.3.15] that there exist an isometry H mapping $R\times R^m$ onto R^{m+1} and a current $Q\in\mathcal{R}_{m-1}^{\text{loc}}(R^m)$ with $D=H_\#(E^1\times Q)$, and from [3, 5.4.8] that Q is absolutely $m-1$ area minimizing with respect to R^m. We note that $\partial Q=0$ because $\partial D=0$. In case $m\geqq 8$ we inductively obtain an open subset Y of R^m such that $Y\cap\operatorname{spt} Q$ is an $m-1$ dimensional submanifold of class ∞ of R^m and $\mathcal{H}^{k-1}(R^m\sim Y)=0$. In case $m=7$ we know from [3, 5.4.15] that $\operatorname{spt} Q$ is a 6 dimensional submanifold of class ∞ of R^7, and we take $Y=R^7$. In both cases $H(R\times Y)\cap\operatorname{spt} D$ is an m dimensional submanifold of class ∞ of R^{m+1} and

$$\mathcal{H}^k[R^{m+1} \sim H(R \times Y)] = \mathcal{H}^k[R \times (R^m \sim Y)] = 0$$

by [3, 2.10.45]. Since $D = \partial(E^{m+1} \llcorner P)$ we see that

$$\Theta^m(\|D\|, x) = 1 \quad \text{for } x \in H(R \times Y) \cap \operatorname{spt} D,$$

hence $\omega(D) \subset \operatorname{spt} D \sim H(R \times Y)$ and $\mathcal{H}^k[\omega(D)] = 0$, which is inconsistent with our previous assertion that $\mathcal{H}^k[\omega(D)] > 0$.

To deduce the conclusion of the theorem from the statement verified above we suppose $a \in R^{m+1} \sim \operatorname{spt} \partial T$ and proceed as in [3, 5.3.18] to find a positive number ρ and a representation

$$T \llcorner U(a, \rho) = \sum_{i \in Z} S_i \quad \text{with} \quad \|T\| \llcorner U(a, \rho) = \sum_{i \in Z} \|S_i\|,$$

where $S_i = [\partial(E^{m+1} \llcorner M_i)] \llcorner U(a, \rho)$ for certain $\mathcal{L}^{m+1}$ measurable sets M_i such that $M_i \subset M_{i-1}$; moreover $\{i : b \in \operatorname{spt} S_i\}$ is finite whenever $b \in U(a, \rho)$. For each integer i we choose an open set W_i such that $W_i \cap \operatorname{spt} S_i$ is an m dimensional submanifold of class ∞ of R^{m+1} and

$$\mathcal{H}^k[U(a, \rho) \sim W_i] = 0 \quad \text{whenever} \quad m - 7 < k \in R.$$

We conclude that $B = U(a, \rho) \sim \bigcup_{i \in Z} (\operatorname{spt} S_i \sim W_i)$ is open,

$$U(a, \rho) \sim B \subset \bigcup_{i \in Z} [U(a, \rho) \sim W_i],$$

$$\mathcal{H}^k[U(a, \rho) \sim B] = 0 \quad \text{whenever} \quad m - 7 < k \in R,$$

$$B \cap \operatorname{spt} T = \bigcup_{i \in Z} B \cap \operatorname{spt} S_i = \bigcup_{i \in Z} B \cap W_i \cap \operatorname{spt} S_i,$$

and $B \cap \operatorname{spt} T$ is an m dimensional submanifold of class ∞ of R^{m+1} because for each $b \in B \cap \operatorname{spt} T$ one can reason as in [3, 5.4.15, p. 646] with a replaced by b to see that $\operatorname{Tan}(\operatorname{spt} T, b)$ is an m dimensional vector space, hence infer from [3, 5.3.18] that b is a regular point for T.

It is not yet known whether the conclusion of Theorem 1 could be sharpened so as to require that $\mathcal{H}^{m-7}(K \sim V) < \infty$ for every compact subset K of $\operatorname{spt} T \sim \operatorname{spt} \partial T$; in case $m = 7$ this holds according to [3, 5.4.16].

3. Next we discuss area minimizing m dimensional chains with arbitrary codimension p in R^{m+p}. When $p > 1$ the singular set can have dimension $m - 2$, as illustrated in [3, 5.4.19] by the example of holomorphic chains. It follows from [3, 5.3.16] that the singular set of an area minimizing m dimensional rectifiable current T is nowhere dense in $\operatorname{spt} T$, but the largest possible value of the dimension of the singular set is not yet known in case $p > 1$ and $m > 1$.

The situation becomes much simpler when Z is replaced as coefficient group by the cyclic group Z_2 of order 2. Reducing modulo 2 in the context of geometric measure theory as explained in [3, 4.2.26], one can modify the proof of Theorem 1 to obtain the following proposition:

THEOREM 2. *If $T \in \mathcal{R}_m(R^{m+p})$ and T is homologically area minimizing modulo 2 with respect to R^{m+p}, which means that $M(T+\partial S+2R) \geq M(T)$ whenever $S \in \mathcal{R}_{m+1}(R^{m+p})$ and $R \in \mathcal{R}_m(R^{m+p})$, then there exists an open set V such that $V \cap \operatorname{spt} T$ is an m dimensional submanifold of class ∞ of R^{m+p} and*

$$\mathcal{H}^k[R^{m+p} \sim (V \cup \operatorname{spt}^2 \partial T)] = 0 \quad \textit{whenever} \quad \sup\{m-2, 0\} < k \in R.$$

In fact the extension of our two lemmas from R^{m+1} to R^{m+p} is trivial, the present current T is representative modulo 2, hence $\omega(T) \sim \operatorname{spt}^2 \partial T$ equals the singular subset of $\operatorname{spt} T \sim \operatorname{spt}^2 \partial T$, and the induction now starts with the case $m=1$ where the singular set is known to be empty.

For $m=2$ it was found in [1, Theorem 3(1)] that the singular set is isolated and $\operatorname{spt} T \sim \operatorname{spt}^2 \partial T$ is the image of an immersion of a 2 dimension manifold in R^{2+p}. However, for $m>2$ it is not yet known whether one could sharpen the conclusion of Theorem 2 so as to require that $\mathcal{H}^{m-2}(K \sim V) < \infty$ for every compact subset K of $\operatorname{spt} T \sim \operatorname{spt}^2 \partial T$.

Recalling [3, 5.4.4] one sees that *Theorem 2 remains valid with R^{m+p} replaced by any $m+p$ dimensional Riemmanian manifold of class ∞.*

For the study of interior regularity of solutions of the problem of least area, use of m dimensional flat chains modulo 2 is substantially equivalent to use of sets with finite m dimensional Hausdorff measure as employed in Reifenberg's approach presented in [4, Chapter 10], provided $G = Z_2$ and L is cyclic (see [4, p. 411]). Then our method shows that the Hausdorff dimension of the singular set of Reifenberg's solution of the m dimensional Plateau problem does not exceed $m-2$.

REFERENCES

1. F. J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) **84** (1966), 277–292. MR **34** #702.

2. E. Bombieri, E. DeGiorgi and E. Giusti, *Minimal cones and the Bernstein theorem*, Invent. Math. **7** (1969), 243–269.

3. H. Federer, *Geometric measure theory*, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, Berlin and New York, 1969.

4. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Die Grundlehren der math. Wissenschaften, Band 130, Springer-Verlag, Berlin and New York, 1966. MR **34** #2380.

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