REPRESENTATIONS OF INFINITE DIMENSIONAL MANIFOLDS AND $\infty - p$ HOMOLOGY FUNCTORS

BY PHILLIP A. MARTENS[1]

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Introduction. The purpose of this note is to announce a representation theorem for separable Fréchet manifolds. This representation theorem demonstrates a close connection between functionals and infinite dimensional spaces. Moreover, it can be applied for the canonical construction of an $\infty - p$ homology functor.

1. **The representation theorem.** Throughout this note $\approx$ will denote homeomorphism or isomorphism, $\sim$ a diffeomorphism, and $\simeq$ a strong homotopy equivalence. Also manifolds are connected.

THEOREM A. *Let E be a separable C^∞ manifold without boundary modeled on the Hilbert space H. Then $\forall p > 0$, $\exists$ an inverse system $\{E_m, p_n^m \mid m \geqq p\}$ with p_n^m onto, E_m open in R^m, and $E \approx$* Inv Lim E_m, *with the standard topology on* Inv Lim E_m. *Also the system $\{E_m\}$ satisfies the additional conditions*:

(a) *$\exists$ connected $m+1$ manifolds with boundary, E_{m+1}^+ and*

$$E_{m+1}^- \ni E_{m+1} = E_{m+1}^+ \cup E_{m+1}^- \quad and \quad E_m = E_{m+1}^+ \cap E_{m+1}^-.$$

(b) $E \simeq$ Dir Lim E_m.

We also have the converse.

THEOREM B. *Given an inverse system $\{E_m, p_n^m \mid m \geqq p\}$, with p_n^m onto and E_m open in R^m, satisfying the following conditions*:

(a) *E_m splits E_{m+1} into sets E_{m+1}^+ and*

$$E_{m+1}^- \ni E_{m+1} = E_{m+1}^+ \cup E_{m+1}^- \quad and \quad E_m = E_{m+1}^+ \cap E_{m+1}^-.$$

(b) Inv Lim E_m *is open in LR^m.*

Then Inv Lim E_m *can be embedded as an open subset of $H \ni E_m$ is embedded as a smooth submanifold. Also* Dir Lim $E_m \simeq$ Inv Lim E_m.

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Details and other applications will appear elsewhere.

2. **Construction of the functor** $H_{\infty-p}(\cdot, Z)$. Alexander [1], Eells [4], Mukherjea [11], and Geba and Granas [6] have introduced techniques to construct $\infty-p$ chains and cochains. The key issue is to construct $\infty-p$ chains with supports in sets of codimension p with a Poincaré duality uniquely determined by the support sets.

The above representations lead to such a construction by allowing sets of codimension p which play a role in the following to be given as inverse limits of sets of dimension $m-p$. (In the following manifolds are without boundary.)

Since in the above representation E_{m-1} splits E_m, we have Mayer-Vietoris exact sequences

$$\to H_{m-p}(\overset{+}{E}_m, Z) \oplus H_{m-p}(\overset{-}{E}_m, Z) \to H_{m-p}(E_m, Z) \underset{m-p}{\longrightarrow} H_{(m-1)-p}(E_{m-1}, Z)$$

which induce the inverse system $\{H_{m-p}(E_m, Z), \partial_{m-p}\}$ so the diagram

$$\begin{array}{ccc} H_{m-p}(E_m, Z) & \overset{\approx}{\leftrightarrow} & H_p(E_m, Z) \\ \partial_{m-p} \downarrow & & \downarrow i^*_{m-1} \\ H_{(m-1)-p}(E_{m-1}, Z) & \overset{\approx}{\leftrightarrow} & H_p(E_{m-1}, Z) \end{array}$$

commutes. Define $H_{\infty-p}(E, Z) = \text{Inv Lim } H_{m-p}(E, Z)$. We then have

THEOREM C. *Let E be an infinite dimensional C^∞ separable Hilbert manifold. Then $\exists$ a homology functor $H_{\infty-p}(\cdot, Z) \ni H_{\infty-p}(E, Z) \approx {}_sH^p(E, Z)$.*

By using the representation theorem for smooth subsets of E of codimension p, local $\infty-p$ chain groups $C_{\infty-p}(U, Z)$ similar to those of Herrera [8] can be constructed so that the induced sheaf structure gives rise to a functor $\mathcal{H}_{\infty-p}(\cdot, Z)$ which leads to the following result.

THEOREM D. *$H_{\infty-p}(E, Z) \approx {}_sH^p(E, Z)$ is canonical in the sense that if* Inv Lim $E_m \approx$ Inv Lim F_m *are two representations for E, then $\exists$ an isomorphism* Inv Lim $H_{m-p}(E_m, Z) \overset{\approx}{\to}$ Inv Lim $H_{m-p}(F_m, Z)$ *so that the composition*

$${}_sH^p(E, Z) \overset{\approx}{\to} \text{Inv Lim } H_{m-p}(E_m, Z) \overset{\approx}{\to} \text{Inv Lim } H_{m-p}(F_m, Z)$$

$$\overset{\approx}{\to} {}_sH^p(E, Z)$$

is the identity.

3. Outline of the proof of Theorem A. The proof of Theorem A follows from the following sequence of results.

THEOREM 1. *Let E be a smooth separable manifold modeled on the Hilbert space H. If $K \subset E$ is closed locally compact subset, then $\exists$ a set $\hat{K} \supset K \ni E \sim E \backslash \hat{K}$.*

The proof of Theorem 1 is an application of the fact that $H \backslash \{0\} \sim H$ [2]. The key to this application is to observe that when K is locally finite dimensional we can choose a $\hat{K} = \bigcup M_i$ where $\{M_i\}$ is a star finite collection of closed finite dimensional submanifolds such that $\hat{K} \supset K$. The map exp then is used to construct a star finite collection of tubular Hilbert neighborhoods $\{N(M_i)\}$ [10], which are trivial since $GL(H)$ is contractible [9]. Using the diffeomorphism $H \backslash \{0\} \sim H$, another diffeomorphism $g_i : E \to E \backslash M_i$ is constructed. Then $g = \prod g_i$ gives the desired map. We can show in the general case that g always operates locally as a product $\prod g_i$.

An application of Morse-Sard approximations [5] gives the following results.

THEOREM 2a. *Let E be a complete separable Hilbert manifold. If $f : E \to R^m$ is a bounded open map and if $\varepsilon : E \to R^+$, then $\exists$ a differentiable $\bar{f}_m : E \to R^m$ which ε approximates f, and with the singular set $S(\bar{f}_m)$ closed and locally compact.*

THEOREM 2b. *The map f_m given by the composition $E \xrightarrow{g} E \backslash \hat{K} \to R^m$, for a suitable $\hat{K} \supset S(\bar{f}_m)$, satisfies the condition that if $e_m, e'_m \in E_m = f_m(E)$, then $\exists$ a bijection $f_m^{-1}(e_m) \to f_m^{-1}(e'_m)$.*

The bijection of Theorem 2b is simply a point-wise bijection. It is achieved by constructing local pseudo-gradient vector fields [13] which are joined along boundaries of charts by taking limits. Then it is possible to follow piecewise smooth pseudo-gradient lines which are joined by the above limits. The only lines which do not transverse every leaf are those whose gradient structure approaches zero. However the union of these lines forms a set which is forced to be closed and locally compact by the structure of $S(\bar{f}_m)$. Hence we can neglect this set by including it in $\hat{K}$.

We can now consider an f given by the composition $E \xrightarrow{\Delta} \prod_1^m E \xrightarrow{j} R^m$, where $j = \prod \bar{d}_{e_i}$, and where $\bar{d}_{e_i}$ is given by the composition $E \sim E \backslash \{e_i\} \to R$, the last map being a metric distance $d(\cdot, e_i)$. The collection $\{e_i\}_{1 \leq i \leq p}$ is chosen to be linearly independent in a suitably small chart containing a point e', and $f_m^{-1}(f_m(e'))$ is a connected leaf of the foliation $\{f_m^{-1}(e_m) = E_e^m \mid E = \bigcup E_e^m\}$.

To prove Theorem A, we will construct the E_m by induction as follows: Given an $f_p: E \to R^p$ satisfying the conditions of Theorem 2b, where E is given a complete metric structure, we construct a corresponding $\bar{f}_1: E_e^p = f_p^{-1}(e_p') \to R^1$. Let $f: E \to R^{p+1}$ be given by $f(e) = (f_p(e), \bar{f}_1(\sigma(e)))$, where σ is the bijection $\sigma: E_e^p \to E_{e'}$. It can be shown that f is differentiable and that

$$\exists \hat{K} \supset S(f) \ni E \overset{g}{\sim} E \backslash \hat{K},$$

so that the composition $e \to g(e) \to Pg(e) \to fPg(e)$ gives a differentiable f_{p+1} without singularities, and where all appropriate pseudo-gradient lines are defined. P is a projection which maps $g(e)$ to a point in E_e^p and is determined by the trivial tubes that produced the g. The projection P has no singularities because we can assume that each $N(M_i)$ is constructed so that g_i operates in directions complementary to $\hat{K}$. In addition, as above $\exists$ a bijection $E_e^{p+1} = f_{p+1}^{-1}(f_{p+1}\ (e)) \to E_{e'}^{p+1}$, for a connected leaf $E_{e'}^{p+1}$. This completes the induction. We can assume:

(a) In a chart about e', every vector in a linearly independent basis transverses some leaf $E_{e'}^m$. Hence $\bigcap_{m \geq p} E_{e'}^m = e'$.

(b) Given a basis of open sets $\{V_{e'}^i\}_{i \geq 1}$

$$f_i^{-1}(V_{e'i}) \subset V_{e'}^i \text{ for some } V_{e'i} \subset E_i = f_i(E).$$

Then taking $E_m = f_m(E)$, we obtain an inverse system $\{E_m, p_n^m \mid m \geq p\}$, where each E_m represents an m-parameter abelian group of bijections, and so that E_m splits E_{m+1}. The above conditions are transported over E by the m-parameter groups E_m. These facts combine to give $E \approx \text{Inv Lim } E_m \supset \text{Inv Lim } R^m$, with Inv Lim E_m open in Inv Lim R^m. The property Dir Lim $E_m \simeq E$ is a form of the Palais-Švarc lemma [12].

Theorem B follows by constructing a homeomorphism $h: H \to$ Inv Lim R^m. Then $h^{-1}(E_m)$ will be a smooth submanifold.

REMARK. Since every Fréchet manifold can be embedded as an open subset of H [7], we can assume that E is any separable Fréchet manifold.

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UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007