REMARKS CONCERNING Ext* (M, $-$)

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Let X be a topological space and let $\mathcal{S}$ (respectively $\mathcal{A}$) be the category of sets (respectively abelian groups). Let $\mathcal{S}'$ (respectively $\mathcal{A}'$) be the category of sheaves of sets (respectively abelian groups) based on X, and fix a sheaf M in $\mathcal{A}'$. The graded functor $\operatorname{Ext}^*(M, -) : \mathcal{A}' \to \mathcal{A}$ is computed as the right derived functors of $\operatorname{Hom}(M, -)$, and of course $\operatorname{Ext}^i(M, N)$ classifies i-fold extensions of M by N [6].

One would also like to be able to classify extensions in nonabelian categories of sheaves. Partial success in this direction has been achieved by Gray [5], but he needs to assume restrictions on X as well as on M. In [10], the author applied triple-theoretic [1] techniques to the category of sheaves of R-algebras (R a sheaf of rings), and successfully classified cohomologically singular extensions of an R-algebra P by one of its modules N.

Specifically, if G is the polynomial algebra cotriple lifted to the category of sheaves of R-algebras, if T is the Godement triple = standard construction [3], and if $\operatorname{Der}_R(P, N)$ is the abelian group of global R-derivations from P to N, then the equivalence classes of singular extensions of P by N are in one-one correspondence with the elements of the first homology group of the double complex $\operatorname{Der}_R(G^*P, T^*N)$. In §II of this note we prove that if G is the free abelian group cotriple lifted to $\mathcal{A}'$ then the nth homology group of the double complex $\operatorname{Hom}(G^*M, T^*N)$ is naturally isomorphic to $\operatorname{Ext}^n(M, N)$. The combination of this theorem and the results in [10] indicates a unified approach to the cohomological classification of extensions in many (algebraic) categories of sheaves.

In §I one can find a theorem which is part of the folklore of triple-theoretic cohomology theory, but for which no straightforward proof appears in print. The theorem is: if an abelian category has an injective cogenerator and E is the model-induced triple then $\operatorname{Ext}^*(M, N)$ and the homology of the complex $\operatorname{Hom}(M, E^*N)$ are naturally isomorphic (note that E is not the triple used by Schafer in [8]).

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The essence of the proof of this theorem appears as a proposition in §I, and we use the proposition again in §II.

I. Ext*(M, $-$) **is a triple-derived functor.** In a number of places (e.g. [7], [4], [6]) one can find a proof of the fact that the category $\mathcal{A}'$ has an injective cogenerator. Thus one can always find an injective resolution I^* for any sheaf N in $\mathcal{A}'$, and $\operatorname{Ext}^n(M, N)$ is defined to be the nth homology group of the complex $\operatorname{Hom}(M, I^*)$.

On the other hand, if I is the injective cogenerator then we can define the "model-induced" triple $\mathbf{E}=(E, \eta, \mu)$ as follows. The functor $E: \mathcal{A}' \to \mathcal{A}'$ is given by $EN=\prod I$ where the product is taken over the set $\operatorname{Hom}(N, I)$. If we write $\langle g \rangle: \prod I \to I$ for the coordinate projection corresponding to the map g in $\operatorname{Hom}(N, I)$ then the natural transformations η, μ are given by $\langle g \rangle \cdot \eta N = g$ and $\langle g \rangle \cdot \mu N = \langle \langle g \rangle \rangle$. Then (E, η, μ) is a triple (see [1]). Moreover, since the product of injectives is injective, EN is injective for each N. We have the complex

$$N \to EN \to E^2N \to E^3N \to \cdots = E^*N$$

where $d: E^kN \to E^{k+1}N$ is $d=\sum_{i=0}^{k} (-1)^i E^i \eta E^{k-i} N$, hence we can consider the homology of the complex $\operatorname{Hom}(M, EN) \to \operatorname{Hom}(M, E^2N) \to \operatorname{Hom}(M, E^3N) \to \cdots$. Denote the nth homology group of this complex by $H^n(M, N)_{\mathbf{E}}$. Then we claim that $H^n(M, N)_{\mathbf{E}} \approx \operatorname{Ext}^n(M, N)$ for all $n \geqq 0$.

LEMMA. *The map $\eta N: N \to EN$ is a monomorphism.*

PROOF. If $f, f': N' \to N$ are such that $\eta N \cdot f = \eta N \cdot f'$ then we must show that $f = f'$. Now for each $g: N \to I$ we have $g \cdot f = \langle g \rangle \cdot \eta N \cdot f = \langle g \rangle \cdot \eta N \cdot f' = g \cdot f'$, and since I is a cogenerator, $f = f'$.

The dual of the following proposition was shown to me by Michael Barr.

PROPOSITION. *If the abelian category $\mathcal{B}$ is endowed with a triple $\mathbf{E}$ such that η is pointwise monic then $N \to E^*N$ is an exact sequence, and conversely.*

PROOF. The converse is obvious. On the other hand, if ηN is monic for each N in $\mathcal{B}$ then we can build an exact sequence

$$0 \to N \to EN \to EC_0 \to EC_1 \to EC_2 \to \cdots = I^*$$

where $C_{-1} = N$ and C_{i+1} is the cokernel of the map $\eta C_i: C_i \to EC_i$ for each $i \geqq -1$. Of course the boundary $EC_i \to EC_{i+1}$ is the composition of the cokernel map $c_{i+1}: EC_i \to C_{i+1}$ and ηC_{i+1}. If $\mathcal{E}$ is the injective class determined by the image of E then any two $\mathcal{E}$-injective and $\mathcal{E}$-exact sequences are homotopic [2]. Now $N \to E^*N$ is $\mathcal{E}$-injective and $\mathcal{E}$-exact. Moreover $N \to I^*$ is $\mathcal{E}$-injective, and we now show that it is also $\mathcal{E}$-exact, i.e. that for any N' we have $\mathrm{Hom}(I^*, EN')$ is exact. Given a cocycle $f: EC_j \to EN'$ we have $0 = df = f \cdot \eta C_j \cdot c_j$ and c_j is epic, hence $f \cdot \eta C_j = 0$. But c_{j+1} is the cokernel of ηC_j and so there is a map $f': C_{j+1} \to EN'$ such that $f' \cdot c_{j+1} = f$. Now the coboundary of $\mu N' \cdot Ef'$ $: EC_{j+1} \to EN'$ is

$$\begin{aligned} d(\mu N' \cdot Ef') &= \mu N' \cdot Ef' \cdot \eta C_{j+1} \cdot c_{j+1} \\ &= \mu N' \cdot \eta EN' \cdot f' \cdot c_{j+1} \\ &= f' \cdot c_{j+1} \\ &= f. \end{aligned}$$

Thus every cocycle is a coboundary, $\mathrm{Hom}(I^*, EN')$ is exact, and I^* is $\mathcal{E}$-exact. It follows that I^* and E^*N are homotopic. But I^* is exact, hence so is E^*N.

COROLLARY. *If* $\mathbf{E}$ *is the model-induced triple on* $\mathcal{A}'$ *defined above then for each* N *in* $\mathcal{A}'$, $N \to E^*N$ *is an injective resolution.*

COROLLARY. $H^*(M, N)_{\mathbf{E}} \approx \mathrm{Ext}^*(M, N)$.

REMARK. The proof works for any abelian category having an injective cogenerator. Dually, if an abelian category has a projective generator and $\mathbf{P}$ is the model-induced cotriple then $H^*(M, N)_{\mathbf{P}} \approx \mathrm{Ext}^*(M, N)$.

II. A double complex yielding $\mathrm{Ext}^*(M, N)$. Consider the following diagram of categories and functors:

$$\begin{array}{ccc} \mathcal{A}' & \underset{Q}{\overset{S}{\rightleftarrows}} & \prod \mathcal{A} \\ F \uparrow\downarrow U & & \prod F_x \uparrow\downarrow \prod U_x \\ \mathcal{S}' & \underset{Q}{\overset{S}{\rightleftarrows}} & \prod \mathcal{S} \end{array}$$

in which the products are taken over all points x in X. S is the stalk functor, i.e. S takes a sheaf to the set of its stalks. Q takes a collection $\{A_x\}$ to the sheaf whose value at an open set V is $\prod A_x$, the product being taken over all points x in V. U and $\prod U_x$ are the obvious "underlying" or "forgetful" functors. F_x is the free abelian group functor. Given a sheaf N in $\mathcal{S}'$, the functor which takes an open set V to the free abelian group on the set $N(V)$ is a presheaf of abelian groups, and FN is defined to be the sheaf associated to this presheaf.

One can show that S is left adjoint to Q, F is left adjoint to U, $\prod F_x$ is left adjoint to $\prod U_x$, $SU\approx\prod U_xS$, and $Q\prod U_x\approx UQ$. Moreover QS is the Godement "standard construction" (see [3]). Let $\mathbf{T}=(T,\eta,\mu)$ be the triple associated to $QS=T$ and $\mathbf{G}=(G,\epsilon,\delta)$ the cotriple associated to $FU=G$ via the adjointnesses. For each M in $\mathfrak{A}'$ we get the complex

$$\cdots\to G^3M\to G^2M\to GM\to M\to 0$$

dually to the way we got $N\to E^*N$ in §I. For each N in $\mathfrak{A}'$ we get the complex $0\to N\to TN\to T^2N\to\cdots$ as in §I. Hence we have the double complex

$$C^{ij}(M,N)=\operatorname{Hom}(G^{i+1}M,T^{j+1}N)\quad\text{for } i,j\geqq 0$$

with boundaries induced by the boundaries in the single complexes. Denote the nth homology group of this double complex by $H^n(M,N)_{\mathbf{G},\mathbf{T}}$.

THEOREM. *$H^n(M,N)_{\mathbf{G},\mathbf{T}}\approx\operatorname{Ext}^n(M,N)$ for each $n\geqq 0$.*

PROOF. It is well known (see [6] or [9]) that $\operatorname{Ext}^*(M,-)$ is a cohomological δ-functor augmented over $\operatorname{Hom}(M,-)$, and that any two such cohomological δ-functors are isomorphic. We thus verify that $H^*(M,-)_{\mathbf{G},\mathbf{T}}$ is such a functor. For convenience we write $H^*(M,-)$ instead of $H^*(M,-)_{\mathbf{G},\mathbf{T}}$.

Given an exact sequence $0\to N'\to N\to N''\to 0$ in $\mathfrak{A}'$ we need to produce an exact triangle

$$H^*(M,N')\to H^*(M,N)\to H^*(M,N'')\xrightarrow{\partial}H^*(M,N').$$

Now T is an exact functor [3] and $\operatorname{Hom}(G^{i+1}M,-)$ is left exact for $i\geqq 0$. Thus

$$0\to\operatorname{Hom}(G^{i+1}M,T^{j+1}N')\to\operatorname{Hom}(G^{i+1}M,T^{j+1}N)$$
$$\to\operatorname{Hom}(G^{i+1}M,T^{j+1}N'')$$

is exact for each $i,j\geqq 0$. Moreover, the last map is onto, for consider the chain of natural isomorphisms:

$$\begin{aligned}\operatorname{Hom}(G^{i+1}M,T^{j+1}N)&\approx\operatorname{Hom}(UG^iM,UT^{j+1}N)\\&\approx\operatorname{Hom}(UG^iM,Q\textstyle\prod U_x(SQ)^jSN)\\&\approx\operatorname{Hom}(SUG^iM,\textstyle\prod U_x(SQ)^jSN)\\&\approx\operatorname{Hom}(SUG^iM,(SQ)^j\textstyle\prod U_xSN).\end{aligned}$$

Since $N\to N''$ is epic, so is $SN\to SN''$ (see [3]) and thus $\prod U_xSN\to\prod U_x\,SN''$ is a split epimorphism. Hence

$$\text{Hom}\,(SUG^iM, (SQ)^j\textstyle\prod U_xSN) \to \text{Hom}\,(SUG^iM, (SQ)^j\prod U_xSN'')$$

is onto. But this map is naturally isomorphic to

$$\text{Hom}\,(G^{i+1}M, T^{j+1}N) \to \text{Hom}\,(G^{i+1}M, T^{j+1}N''),$$

which is therefore onto.

It follows that $0 \to C^{ij}(M, N') \to C^{ij}(M, N) \to C^{ij}(M, N'') \to 0$ is exact for each $i, j \geqq 0$ and that $0 \to C^{**}(M, N') \to C^{**}(M, N) \to C^{**}(M, N'') \to 0$ is an exact sequence of double complexes. The exact homology triangle is now a standard result of homological algebra. Hence $H^*(M, -)$ is an exact δ-functor.

The proof is completed by showing that $H^*(M, -)$ is augmented over $\text{Hom}(M, -)$ and that $H^n(M, -)$ is effaceable for each $n > 0$. First, $H^0(M, N)$ is the intersection of the kernels of the two maps $C^{0,0}(M, N) \to C^{0,1}(M, N)$ and $C^{0,0}(M, N) \to C^{1,0}(M, N)$. Now $\epsilon M: GM \to M$ is an epimorphism (essentially because the associated sheaf functor is exact) and $\eta N: N \to TN$ is a monomorphism [3]. Hence by the proposition in §I and its dual, $N \to T^*N$ and $G^*M \to M$ are exact sequences. But $\text{Hom}(-, -)$ is left exact and thus

$$\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \text{Hom}\,(M, N) & \to & \text{Hom}\,(GM, N) & \to & \text{Hom}\,(G^2M, N) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \text{Hom}\,(M, TN) & \to & \text{Hom}\,(GM, TN) & \to & \text{Hom}\,(G^2M, TN) \\
 & \downarrow & & \downarrow & & \\
0 \to & \text{Hom}\,(M, T^2N) & \to & \text{Hom}\,(GM, T^2N) & &
\end{array}$$

is exact. This implies that $H^0(M, N) \approx \text{Hom}(M, N)$ and $H^*(M, -)$ is augmented over $\text{Hom}(M, -)$.

Finally, to demonstrate the effaceability, let N be injective in $\mathfrak{A}'$. Then $\eta N: N \to TN$ is a split monomorphism, say $u \cdot \eta N = N$. As is shown in [1], the maps $T^n u$ provide a contraction of the complex $0 \to N \to T^*N$. Thus for each $j \geqq -1$ the column $C^{j*}(M, N)$ is exact and has zero homology. It follows that the total homology of $C^{**}(M, N)$ vanishes in positive dimensions, that is, $H^n(M, N) = 0$ if $n > 0$ and if N is injective. Hence $H^n(M, -)$ is effaceable for $n > 0$. This completes the proof of the fact that $H^*(M, -)$ is a cohomological δ-functor augmented over $\text{Hom}(M, -)$, which was to be shown.

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