GROWTH RATE OF GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS

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Communicated by David Blackwell, November 3, 1969

1. Statement of results. Let $(Y,{}_t, t \geqq 0)$ be a real, separable Gaussian process with stationary increments, mean 0, and $Y_0=0$. Let $2Q(t)$ be the variance of Y_t and define

$$X_t = Y_t/(2Q(t))^{1/2}.$$

THEOREM 1. *Suppose there exists a nonnegative function $v(t)$ such that*

$$(*) \qquad \lim_{t\to\infty} \frac{Q(s+t)-Q(s)}{v(s+t)-v(s)} = 1 \quad \textit{uniformly in } s$$

and there exist positive constants s_0, β_1, β_3 with $1 \leqq \beta_3 \leqq (\beta_1/2+1)$ such that

(i) *is monotone nondecreasing,*
(ii) $v(\lambda s) \geqq \lambda^{\beta_1} v(s) > 0, \qquad s \geqq s_0, \lambda \geqq 1,$
(iii) $v(\lambda s) \leqq \lambda^{\beta_3} v(s), \qquad s \geqq s_0, \lambda \geqq 1$

and suppose that there exists $\beta_2 > 0$ such that

(iv) $Q(t) = O(t^{\beta_2}), \qquad t \downarrow 0.$

Then

$$\limsup_{t\to\infty}(X_t - (2 \log\log t)^{1/2}) = 0 \quad a.s.$$

In fact somewhat more is true.

THEOREM 2. *Under the assumptions of Theorem* 1,

$$\lim_{T\to\infty} \left(\sup_{t \leqq T} X_t - (2 \log\log T)^{1/2}\right) = 0 \quad a.s.$$

An important class of examples is obtained by taking $Y_t = \int_0^t Y_s' ds$ where (Y_s') is a real stationary Gaussian process with mean 0 and continuous sample functions. If $q(|t-s|)$ is the covariance of the (Y_t)-process and $R(t) = \int_0^t (q(s)ds$ then $Q(t) = \int_0^t R(s)ds$. If $v(t)$ is a differentiable function satisfying conditions (i), (ii) and (iii) of The-

AMS Subject Classifications. Primary 6030, 6069.

Key Words and Phrases. Gaussian processes, stationary increments, law of the iterated logarithm, limiting distribution of the maximum.

orem 1 and $R(t)\sim v'(t)$ as $t\to\infty$, then (*) will hold, and condition (iv) will hold with $\beta_2=2$. In particular if $R(t)$ tends to a positive constant K, the hypotheses of the theorem will be satisfied with $v(t)=Kt$. This will be the case if $q(t)$ is always nonnegative and tends to zero sufficiently rapidly as t approaches infinity, e.g. when (Y'_t) is the stationary Ornstein-Uhlenback process with covariance $e^{-|t-s|}$.

2. **Idea of proof.** Note that the covariance of (Y_t), hence that of (X_t) depends only on the function Q. In fact

$$E[Y_sY_t] = Q(s) + Q(t) - Q(t-s).$$

The results are reduced to the following theorem on stationary Gaussian processes; the proof is given in [1].

THEOREM (PICKANDS). *Let $(Z_t, t\geqq 0)$ be a real, separable, stationary Gaussian process, with mean* 0 *and covariance $r(|t-s|)$ such that for some $\alpha>0$,*

$$1 - r(t) = O(t^\alpha), \qquad t \downarrow 0$$

and

$$r(t) = O((\log t)^{-1}), \qquad t \to \infty.$$

Then

$$\lim_{T\to\infty}\left(\sup_{t\leqq T} Z_t - (2\log T)^{1/2}\right) = 0.$$

The plan is to compare $X_t^* = X_{e^t}$ with stationary processes to which the theorem of Pickands is applicable. In [2] the following important result is proved.

LEMMA (SLEPIAN). *Let X_j^-, X_j^+, $j=1, 2 \cdots N$ be two Gaussian sequences of random variables, with mean* 0 *and covariances $r^-(i,j)$, $r^+(i,j)$ respectively, $1\leqq i\leqq N$, $1\leqq j\leqq N$. Suppose*

$$r^-(i,i) = r^+(i,i),\ i = 1, 2 \cdots N; \qquad r^-(i,j) \leqq r^+(i,j),\ 1 \leqq i \leqq j \leqq N.$$

Then for any choice of constants a_j, $j=1, 2, \cdots N$

$$P\left[\bigcap_{j=1}^{N} [X_j^- \leqq a_j]\right] \leqq P\left[\bigcap_{j=1}^{N} [X_j^+ \leqq a_j]\right].$$

The argument essentially consists of showing that the hypotheses of Theorem 1 permit construction of stationary processes (X_t^-), (X_t^+) satisfying the conditions of the theorem of Pickands and having covariances related to that of (X_t^*) in such a manner that Slepian's lemma allows one to deduce that (X_t^*) also satisfies the conclusion of Pickands's theorem.

References

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