MEASURES WHICH ARE CONVOLUTION EXPONENTIALS[1]

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Let $M(R)$ denote the measure algebra on the additive group of the reals. R. G. Douglas recently pointed out to us the importance of the following question in the study of Wiener-Hopf integral equations: if $\mu \in M(R)$ is invertible, then under what conditions does $\mu = \exp(\nu)$ for some $\nu \in M(R)$?

The relevance of the above question in integral equations stems from the fact that if $\mu \in M(R)$ is invertible, then μ is an exponential if and only if μ has a factorization of the form $\mu = \mu_1 * \mu_2$, where μ_1 and μ_2 are invertible elements of $M[0, \infty)$ and $M(-\infty, 0]$ respectively. In fact, if $\mu = \exp(\nu)$ and $\nu_1 = \nu|_{[0,\infty)}$, $\nu_2 = \nu|_{(-\infty,0)}$, then $\mu_1 = \exp(\nu_1)$ and $\mu_2 = \exp(\nu_2)$ yields such a factorization.

Now if W_μ is the Wiener-Hopf operator on $L^p[0, \infty)$ $(p \geqq 1)$ given by

$$W_\mu f(x) = \int_0^\infty f(y) d\mu(x - y), \tag{1}$$

then it is easy to see that W_μ is invertible if $\mu = \mu_1 * \mu_2$ with μ_1 and μ_2 invertible elements of $M[0, \infty)$ and $M(-\infty, 0]$ respectively. In fact, $W_\mu = W_{\mu_2} \circ W_{\mu_1}$, $W_{\mu_1}^{-1} = W_{\mu_1^{-1}}$, and $W_{\mu_2}^{-1} = W_{\mu_2^{-1}}$ in this case (however, it may not be true that $W_\mu = W_{\mu_1} \circ W_{\mu_2}$). Thus, if μ is an exponential, W_μ is an invertible Wiener-Hopf operator. A general survey of the invertibility problem for Wiener-Hopf operators appears in [3].

If A is a commutative Banach algebra with identity, let A^{-1} and $\exp(A)$ denote the group of invertible elements of A and the subgroup consisting of the range of the exponential function. It is well known that $\exp(A)$ is the connected component of the identity in A^{-1}. The index group of A is the factor group $A^{-1}/\exp(A)$. Arens [1] and Royden [5] have shown that this is isomorphic to the first Čech cohomology group, with integral coefficients, of the maximal ideal space of A. Our problem then, is to determine the index group of

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$M(R)$, or more precisely, to determine for $\mu \in M(R)^{-1}$ the class of μ in $M(R)^{-1}/\exp(M(R))$. We do not have a complete solution to this problem, but we shall report on results which significantly extend prior knowledge in this direction. Details will appear elsewhere in a paper investigating the cohomology of the maximal ideal space of a general measure algebra.

If $\mu \in M(R)^{-1}$ has the form $\mu = \lambda\delta_0 + \nu$ with $\lambda \in \mathbf{C}$, δ_0 the point mass at zero, and ν absolutely continuous, then it is well known that $\mu \in \exp(M(R))$ if and only if the winding number about zero of the Fourier transform $\hat{\mu}$ of μ is zero ($\hat{\mu}$ is considered a function on the one-point compactification of R—i.e., on the circle). This is equally true if it is only assumed that $\nu \in (L^1(R))^{1/2}$—the intersection of all maximal ideals of $M(R)$ containing $L^1(R)$. Thus, winding number provides an isomorphism between the index group of the algebra $\mathbf{C}\delta_0 + (L^1(R))^{1/2}$ and the integers.

If $\mu \in M(R)^{-1}$ is a discrete measure, then Bohr [2] proved that $\mu = \delta_c * \exp(\nu)$ for a unique real number c and some discrete measure ν. The number c is called the mean motion of the almost periodic function $\hat{\mu}$. The correspondence $\mu \to c$ induces an isomorphism between the index group of the algebra of discrete measures and the group of reals.

Our main theorem extends Bohr's result. It says that the index of a discrete measure is not changed by the addition of a sufficiently singular continuous measure.

Let $|\mu|$ denote the total variation of $\mu \in M(R)$. If each convolution power $|\mu|^n$ $(n \geqq 0)$ is purely singular, then we shall say that μ is permanently singular.

THEOREM 1. *If $\mu \in M(R)^{-1}$ is permanently singular, then there is a unique real number c so that $\mu = \delta_c * \exp(\nu)$ for some $\nu \in M(R)$. The number c is the mean motion of the Fourier transform of the discrete part of μ.*

COROLLARY. *If $\mu \in M(R)^{-1}$ is permanently singular, then for some real number c, $\mu = \delta_c * \mu_1 * \mu_2$, where $\mu_1 \in M[0, \infty)^{-1}$ and $\mu_2 \in M(-\infty, 0]^{-1}$.*

Unfortunately, the class of permanently singular measures does not seem to be closed under either addition or multiplication. Every measure in $M(R)$ has a decomposition of the form

$$\mu = \omega + \sum_{i=1}^{\infty} \nu_i \tag{2}$$

with $\omega \in (L^1(R))^{1/2}$ and ν_i permanently singular for each i. However, if more than one of the ν_i's is nonzero this decomposition does not help in the computation of the index of μ. If $\mu=\omega+\nu$ with $\omega\in(L^1(R))^{1/2}$ and ν permanently singular, then $\mu\in M(R)^{-1}$ implies $\nu\in M(R)^{-1}$ and $\nu=\delta_c * \exp(\rho)$, by Theorem 1. Hence, $\mu=\delta_c * \exp(\rho) * \mu'$, where $\mu'=\delta_0+\delta_{-c} * \exp(-\rho) * w$ is an element of $\mathbf{C}\delta_0+(L^1(R))^{1/2}$. It follows that $\mu\in\exp(M(R))$ if and only if $c=0$ and $\hat{\mu}'$ has winding number zero. An attempt to apply this procedure when the decomposition of μ in (2) contains more than one ν_i leads to a possibly nonconvergent infinite product. Hence the problem remains open for general μ.

On the proof of Theorem 1. Our proof of Theorem 1 is based on the structure theory for convolution measure algebras (cf. [6], [7], [8]). If T is a locally compact topological semigroup, then any L-subalgebra of the measure algebra $M(T)$ is a convolution measure algebra. An L-subalgebra of $M(T)$ is a closed subalgebra $\mathfrak{M}$ such that $\mu\in\mathfrak{M}$, $\nu\in M(T)$, and ν absolutely continuous with respect to μ imply that $\nu\in\mathfrak{M}$.

The main theorem of [6] states that a commutative semisimple convolution measure algebra $\mathfrak{M}$ can be represented as a weak-*dense L-subalgebra of $M(S)$ for a compact abelian semigroup S, in such a way that every complex homomorphism h of $\mathfrak{M}$ has the form $h(\mu)=\int f d\mu$ for some continuous semicharacter on S. The semigroup S is called the structure semigroup of $\mathfrak{M}$.

Let $\mathfrak{M}$ be a commutative semisimple convolution measure algebra with a normalized identity. Then S has an identity and its space of nontrivial continuous semicharacters, $\hat{S}$, is a semigroup under pointwise multiplication. If $\hat{S}$ is given the Gelfand or weak topology induced by $\mathfrak{M}$, then it can be identified with the maximal ideal space of $\mathfrak{M}$. With this topology $\hat{S}$ is a compact semitopological semigroup (i.e., multiplication is only separately continuous). If multiplication in $\hat{S}$ were jointly continuous, then the structure theory of compact topological semigroups (cf. [4]) would make the cohomology of $\hat{S}$ very computable. Unfortunately, this is not the case. However, results of [7] allow us to circumvent this difficulty and obtain the results described below.

Lemma 1. *Let $\mathfrak{M}$ be a semisimple commutative convolution measure algebra with normalized identity δ. Let $\mathfrak{M}$ satisfy the following two conditions:*

(1) *there is $\mu\in\mathfrak{M}$ such that every $\nu\in\mathfrak{M}$ is absolutely continuous with respect to μ;*

(2) *if an L-subalgebra $\mathfrak{N}$ of $\mathfrak{M}$ is isomorphic to a group algebra $L^1(G)$, then $\mathfrak{N}$ contains the identity δ.*

Then the natural map of $\hat{S}$ onto its kernel (minimal ideal) is a deformation retract.

In the presence of condition (1) above, there is a natural way of trying to construct the homotopy guaranteed by Lemma 1. It turns out that the only obstructions to completing the process are due to group algebras in $\mathfrak{M}$ which do not contain the identity.

Since every convolution measure algebra is an inductive limit of algebras which satisfy condition (1), the following theorem follows directly from Lemma 1 and the continuity properties of Čech cohomology:

THEOREM 2. *If $\mathfrak{M}$ is a semisimple commutative convolution measure algebra satisfying condition* (2) *of Lemma* 1, *then the natural map of $\hat{S}$ onto its kernel induces an isomorphism of Čech cohomology.*

Returning to Theorem 1, if $\mu \in M(R)^{-1}$ is permanently singular, then one can show that there is an L-subalgebra $\mathfrak{M}$ of $M(R)$ which contains μ, μ^{-1}, and all discrete measures, but is disjoint from $L^1(R)$. It follows that $\mathfrak{M}$ satisfies the hypothesis of Theorem 2. Furthermore, the kernel of $\hat{S}$ in this case is the maximal ideal space of the algebra of discrete measures. Hence, Theorem 1 follows from Theorem 2 and the Arens-Royden Theorem.

ADDED IN PROOF. We can now prove that if $\mu \in M(R)$ then $\mu = \nu * \exp(\omega) * \delta_c$, where $\nu \in L^1(R) + \delta_0$, $\omega \in M(R)$, and $c \in R$. This implies that $M(R)^{-1}/\exp(M(R)) \approx Z \oplus R$.

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