TWO REMARKS ON A. GLEASON'S FACTORIZATION THEOREM

BY GIOVANNI VIDOSSICH

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The theorem of A. Gleason [2, vii.23] asserts that every continuous map f from an open subset U of a product X of separable topological spaces into a Hausdorff space Y whose points are G_δ-sets has the form $g \circ \pi|_U$, where π is a countable projection of X and $g: \pi(U) \to Y$ is continuous. A natural question is to find what other "pleasant" subsets U of X have the above factorization property. The most plausible ones are compact subsets: for, if $U \subseteq X$ is compact and $f = g \circ \pi|_U$ with f continuous, then g must be continuous since $\pi|_U$ is a closed map (being continuous on a compact space).

The first part of this note rejects this conjecture by giving an example of a compact subset of a product of copies of the unit interval, without the factorization property. In the second part, it is proved that the factorization $f = g \circ \pi|_U$ always holds whenever f is uniformly continuous and the range metric. This result implies an open mapping theorem for continuous linear mappings on products of Fréchet spaces.

1. **The example.** Let Z be a compact Hausdorff space which is first countable but not metrizable. Such a space exists by [1, §2, Exercise 13]. Since Z is completely regular, Z is homeomorphic to a compact subset U of a product X of copies of $[0, 1]$. Let $f: U \to U$ be the identity. Assume that $f = g \circ \pi|_U$, with π a countable projection and $g: \pi(U) \to U$ continuous, and argue for a contradiction. Since countable products of separable metric spaces are separable metric, $\pi(U)$ is separable metric. Hence U is a continuous image of a separable metric space. But a cosmic metric space is metrizable whenever it is compact by [3, p. 994, (C) for cosmic spaces]. This contradicts the assumptions on Z.

2. **A factorization theorem.** The above example shows that the following result does not hold longer when Y is not metrizable.

THEOREM. *If Z is any subset of a product of arbitrary uniform spaces*

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X_α ($\alpha \in A$) into a metric space Y, then every uniformly continuous f: $Z \to Y$ has the form $g \circ \pi|_Z$ with π a countable projection and g uniformly continuous.

PROOF. By the uniform continuity of f, for each integer $n \geqq 1$ there are a finite subset $A_n \subseteq A$ and uniform covers $\mathfrak{U}_\alpha$ of X_α ($\alpha \in A_n$) such that

$$d(f(x), f(y)) \leqq 1/n$$

whenever x, $y \in Z$ have the coordinates corresponding to $\alpha \in A_n$ near of order $\mathfrak{U}_\alpha$. Put $C = \bigcup_{n=1}^\infty A_n$ and π the countable projection $(x_\alpha)_{\alpha \in A} \to (x_\alpha)_{\alpha \in C}$. For every $x \in \pi(Z)$, let z_x be a point of $Z \cap \pi^{-1}(x)$. Define g: $\pi(Z) \to Y$ by $x \rightsquigarrow f(z_x)$. If z', $z'' \in Z$ have the same image by π, then $d(f(z'), f(z'')) \leqq 1/n$ for all $n \geqq 1$ (since $C \supseteq A_n$), which implies $d(f(z'), f(z'')) = 0$, i.e. $f(z') = f(z'')$. This means that g is well defined. From the definition it follows $f = g \circ \pi|_Z$. The equality $f = g \circ \pi|_Z$ means that two points of Z have the same image by f whenever they have the same coordinates for $\alpha \in C$. By this and $C \supseteq A_n$ ($n \geqq 1$), g is uniformly continuous. Q.E.D.

COROLLARY. *Let X_α ($\alpha \in A$), Y be arbitrary complete metrizable topological vector spaces. Then every continuous linear map f from $\prod_{\alpha \in A} X_\alpha$ onto Y is open.*

PROOF. Since a continuous linear map is uniformly continuous in the standard uniformities of topological vector spaces, the above theorem implies that $f = g \circ \pi$, with π a countable projection and g uniformly continuous. Since π and f are linear, g is linear. Since a countable product of complete metric spaces is complete metric, g is open by Banach homomorphism theorem. Since π is open, f must be also. Q.E.D.

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UNIVERSITY OF PISA, ITALY