# BANACH SPACES OF LIPSCHITZ FUNCTIONS AND VECTOR-VALUED LIPSCHITZ FUNCTIONS[1]

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**Introduction.** Given a metric space $(S, d)$, and a Banach space $E$, let $\mathrm{Lip}_E(S, d)$ denote the vector space of bounded functions $f: S \to E$ such that

$$\|f\|_d = \sup\{\|f(s) - f(t)\| d^{-1}(s, t) \mid s \neq t\}$$

is finite. Let $\|\cdot\|_\infty$ denote the sup-norm. Then $\mathrm{Lip}_E(S, d)$ endowed with $\|\cdot\| = \max(\|\cdot\|_\infty, \|\cdot\|_d)$ is a Banach space. $\mathrm{lip}_E(S, d)$ denotes the closed subspace of functions $f$ such that

$$\lim_{d(s,t)\to 0} \|f(s) - f(t)\| d^{-1}(s, t) = 0.$$

When $E$ is the set of real or complex numbers, we drop the subscript and write $\mathrm{Lip}(S, d)$ and $\mathrm{lip}(S, d)$.

In this paper we examine the Banach space properties of $\mathrm{lip}(S, d)$ and $\mathrm{Lip}(S, d)$ and extend some known results. Details and proofs of results presented here will appear elsewhere.

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## 1. Weak completeness and extreme points.

THEOREM 1.1. *Let $(S, d)$ be any metric space. A sequence $\{f_n\}$ in $\mathrm{lip}(S, d)$ is weakly Cauchy if and only if it is bounded and every sequence $\{s_m\}$ in $S$ has a subsequence $\{s_{m_i}\}$ such that* $\lim_{n\to\infty} \lim_{i\to\infty} f_n(s_{m_i})$ *exists.*

COROLLARY 1.2. *If $(S, d)$ is compact, $\{f_n\}$ is weakly Cauchy if and only if it is bounded and* $\lim_{n\to\infty} f_n(s)$ *exists for each $s \in S$.*

If $0 < \alpha \leq 1$ and $d$ is a metric, so is $d^\alpha$. We frequently consider $\mathrm{lip}(S, d^\alpha)$ for $0 < \alpha < 1$, since this space separates points.

THEOREM 1.3. *Let $(S, d)$ be any metric space and $0 < \alpha < 1$. Then neither* $\mathrm{Lip}(S, d^\alpha)$ *nor* $\mathrm{lip}(S, d^\alpha)$ *is weakly sequentially complete unless $S$ is a finite set.*

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The proof of 1.3 consists in using 1.1 to show that if $\mathrm{lip}(S, d^\alpha)$ is weakly sequentially complete, it is isomorphic with the space of bounded functions on $S$.

In [6] Jenkins characterized the extreme points of the dual ball of $\mathrm{lip}(S, d^\alpha)$ when $0<\alpha<1$ and $(S, d)$ is compact. This generalized a result of deLeeuw [3]. He also gave a partial description of the extreme points of the dual ball of $\mathrm{Lip}(S, d^\alpha)$. We state this latter result below for reference.

If $f\in \mathrm{Lip}(S, d^\alpha)$ and $W=\{(s, t)\in S\times S \mid s\neq t\}$, define $\tilde{f}(s, t)=(f(s)-f(t))/d^\alpha(s, t)$ if $(s, t)\in W$ and $\tilde{f}(s)=f(s)$ if $s\in S$. Then $\tilde{f}\in C(S\cup W)$, the bounded continuous functions on $S\cup W$. Let $S$ be compact. We can extend each $\tilde{f}$ to an element $\hat{f}$ in $C(S\cup\beta W)$, where $\beta W$ is the Stone-Čech compactification of $W$. $f\to\hat{f}$ is a linear isometry. The result of Jenkins states that the extreme points of the dual ball are precisely the functionals of the form $f\to\lambda f(s)$ and $f\to\lambda\tilde{f}(s, t)$, for $s\in S$, $0<d(s, t)<2$ and $|\lambda|=1$, plus some subset $Q$ of functionals of the form $f\to\lambda\hat{f}(\omega)$ where $\omega\in\beta W\sim W$, $|\lambda|=1$. The nature of $Q$ was left an open problem in [6]. Using 1.3 and Rainwater's theorem (see [7]), we get

THEOREM 1.4. *$Q\neq\emptyset$ unless $S$ is a finite set.*

## 2. Compactness.

LEMMA 2.1. *Let $(S, d)$ be any metric space and $F$ a bounded subset of* $\mathrm{Lip}(S, d)$. *Then $\tilde{F}=\{\tilde{f} \mid f\in F\}$ is equicontinuous at each point of $S\cup W$.*

THEOREM 2.2. *Let $(S, d)$ be compact. A subset $F$ of* $\mathrm{lip}(S, d)$ *is relatively compact if and only if*

$$\lim_{d(s,t)\to 0} d^{-1}(s, t)\,|f(s)-f(t)| = 0 \quad \textit{uniformly for } f\in F.$$

COROLLARY 2.3. *The unit ball of* $\mathrm{Lip}(S, d^\beta)$ *is compact in* $\mathrm{lip}(S, d^\alpha)$ *for* $0<\alpha<\beta\leq 1$.

THEOREM 2.4. *The following are equivalent:*

(a) *$(S, d)$ is precompact.*

(b) *The unit ball of* $\mathrm{Lip}(S, d^\alpha)$, $0<\alpha\leq 1$, *is compact for the sup-norm topology.*

(c) *The unit ball of* $\mathrm{Lip}(S, d^\beta)$ *is compact in* $\mathrm{lip}(S, d^\alpha)$ *for* $0<\alpha<\beta\leq 1$.

REMARK. A characterization of the weakly relatively compact subsets of $\mathrm{lip}(S, d)$ similar to 2.2 but involving quasi-equicontinuity can be given.

3. $\mathrm{Lip}_{E'}(S, d)$ **as a dual space and** $\mathrm{Lip}(S, d^\alpha)$ **as a second dual.** Several results have appeared which recognize various spaces of Lipschitz functions as dual spaces (see [1] and [8]). By using a theorem due to Dixmier [4], we obtain the following result.

THEOREM 3.1. *For any metric space* $(S, d)$ *and any Banach space* $E$, $\mathrm{Lip}_E(S, d)$ *is a dual space whenever* $E$ *is.*

Considering $\mathrm{Lip}(S, d)$ and $\mathrm{Lip}_E(S, d)$ as dual spaces, we can characterize weak* sequential convergence.

THEOREM 3.2. *A sequence* $\{f_n\}$ *in* $\mathrm{Lip}_E(S, d)$ *converges weak* to zero if and only if it is bounded and* $\{f_n(s)\}$ *converges weak* to zero in* $E$ *for each* $s \in S$. *If* $(S, d)$ *is compact, a sequence* $\{g_n\}$ *converges weak* to zero in* $\mathrm{Lip}(S, d)$ *if and only if it is bounded and converges uniformly to zero.*

For the rest of this paper, unless otherwise stated, assume that every closed bounded subset of $(S, d)$ is compact. Let $\mathrm{lip}^0(S, d^\alpha)$ denote the space of functions in $\mathrm{lip}(S, d^\alpha)$ that vanish at infinity. The main result of [6] is that for real-valued functions, the bidual of $\mathrm{lip}^0(S, d^\alpha)$ $(0<\alpha<1)$ is $\mathrm{Lip}(S, d^\alpha)$. For complex-valued functions, an extra hypothesis on $(S, d)$ was required. (This generalized the original work of deLeeuw [3] along these lines.) By using 3.1, we are able to give a different proof of this result which shows the extra hypothesis in the complex case to be unnecessary.

4. $\mathrm{Lip}_{E''}(S, d)$ **as a second dual.** Given a Banach space $E$, let $E'$ and $E''$ denote its dual and bidual respectively. Since $\mathrm{Lip}_{E''}(S, d^\alpha)$ is a dual space, it is natural to ask if it is the bidual of $\mathrm{lip}_E^0(S, d^\alpha)$, $0<\alpha<1$. ($f: S\to E$ is said to vanish at infinity if $|f|$ does, where $|f|(s) = \|f(s)\|$).

In the scalar-valued case the canonical mapping of $\mathrm{lip}^0(S, d^\alpha)''$ into $\mathrm{Lip}(S, d^\alpha)$ is shown to be one-to-one by showing that the point evaluations span a dense subspace of $\mathrm{lip}^0(S, d^\alpha)'$. We are able to extend this to the vector-valued case by defining a canonical map $\Lambda: \mathrm{lip}_E^0(S, d^\alpha)'' \to \mathrm{Lip}_{E''}(S, d^\alpha)$ in an analagous way and showing that functionals of the form $f \to \langle f(s), x'\rangle$, $s\in S$, $x'\in E'$, $f \in \mathrm{lip}_E^0(S, d^\alpha)$, span a dense subspace of $\mathrm{lip}_E^0(S, d^\alpha)'$. This is done by use of vector-valued measures.

At this point, the methods used in the scalar-valued case fail because of an inability to extend vector-valued Lipschitz functions in a norm-preserving way. To bypass this, the theory of tensor products of Banach spaces is employed. The following result is vital for this purpose.

THEOREM 4.1. *For any metric space* $(S, d)$ *and Banach space* $E$, $\text{Lip}_{E'}(S, d)$ *is isometrically isomorphic with the space* $\mathfrak{L}(E, \text{Lip}(S, d))$ *of bounded linear maps from* $E$ *into* $\text{Lip}(S, d)$.

REMARK. When $(S, d)$ is compact, the isometry in 4.1 identifies $\text{lip}_{E'}(S, d)$ with the compact operators having range in $\text{lip}(S, d)$.

Now, if $V$ denotes the closed linear span in $\text{Lip}(S, d)'$ of the point evaluations, then the tensor product $E \otimes V$ of $E$ and $V$ can be canonically identified (algebraically) with a subspace of $\text{Lip}_{E'}(S, d)'$ by $x \otimes \phi(f) = \phi(x \circ f)$, where $x \circ f$ is the element of $\text{Lip}(S, d)$ defined by $x \circ f(s) = \langle x, f(s) \rangle$ for $s \in S$.

From 4.1 and a theorem due to Schatten [10, Theorem 3.2, p. 47] we obtain the basic

THEOREM 4.2. *Let* $(S, d)$ *be any metric space and* $E$ *any Banach space. Then the norm induced on* $E \otimes V$ *as a subspace of* $\text{Lip}_{E'}(S, d)'$ *is the greatest crossnorm* $\gamma$.

Let $\lambda$ denote the so-called least crossnorm and $\lambda'$ the dual norm.

THEOREM 4.3. *The norms* $\lambda'$ *and* $\gamma$ *agree on* $E' \otimes \text{lip}^0(S, d^\alpha)'$ *if and only if* $\Lambda$ *is a surjective isometry and* $E \otimes \text{lip}^0(S, d^\alpha)$ *is dense in* $\text{lip}_E^0 (S, d^\alpha)$ *(under the usual identification* $x \otimes f \to f \cdot x$*).*

Now, a theorem due to Grothendieck (see [5, Corollary 5.2, p. 110]) gives

THEOREM 4.4. *Let* $E$ *be a Banach space, and* $0 < \alpha < 1$. *If either* $E'$ *or* $\text{lip}^0(S, d^\alpha)$ *has the approximation property (see* [9, Chapter 10, §9]*), then* $\Lambda$ *is a surjective isometry.*

When $(S, d)$ is the unit interval, $\text{lip}(S, d^\alpha)'$ is isomorphic with $l_1$ (see [2]) and therefore has the approximation property. It is not known whether this is true in general.

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