TWO L^p INEQUALITIES[1]

BY CATHLEEN S. MORAWETZ

Communicated by M. H. Protter, June 30, 1969

We present here two new inequalities for the space of vector-valued functions X in L^p, $p>1$ with the norm $\|X\|$ satisfying $\|X\|^p = \int |X|^p d\mu$. The inequalities are extensions of those given by K. O. Friedrichs [1] and can be used respectively instead of Clarkson's inequality, [2], to give simple proofs that L^p space is uniformly convex (rotund) and uniformly smooth. A different proof of the uniform convexity was given by Beurling in a lecture and for $p \geqq 2$ by Mostow [3]. For earlier results on the uniform smoothness see Day [4].

The two inequalities (global) are for $p>1$,

$$\text{(Ia)} \qquad \frac{\|X\|^p + \|Y\|^p}{2} - \left\|\frac{X+Y}{2}\right\|^p \geqq a\left\|\frac{X-Y}{2}\right\|^{p/s}\left(\frac{\|X\|^p+\|Y\|^p}{2}\right)^{1-(1/s)}$$

where $a=a(p)>1$, $s=1$ for $p<2$, $s=p/2$ for $p>2$, and

$$\text{(Ib)} \qquad \left(\tfrac{1}{2}\|X\|^p + \tfrac{1}{2}\|Y\|^p\right) \leqq \left\|\frac{X+Y}{2}\right\|^p\left(1 + b_1\left(\frac{\|X-Y\|}{\|X+Y\|}\right)^s + b_2\left(\frac{\|X-Y\|}{\|X+Y\|}\right)^p\right)$$

where $b_1=b_1(p)$, $s=2$, vanishes for $p\leqq 2$ and $b_2=b_2(p)$. Note by convexity since $p>1$,

$$\text{(1)} \qquad \left\|\frac{X+Y}{2}\right\| \leqq \tfrac{1}{2}(\|X\|+\|Y\|) \leqq \left(\tfrac{1}{2}\|X\|^p + \tfrac{1}{2}\|Y\|^p\right)^{1/p} \leqq \max(\|X\|, \|Y\|).$$

We set $X+Y=2A$, $X-Y=2D$ and introduce $r=\|D\|/\|A\|$ and $m=(\frac{1}{2}\|X\|^p+\frac{1}{2}\|Y\|^p)^{1/p}$. Then one notes that the two inequalities may be used to confine the ratio $\|A\|/m$ in the form

$$(1+b_1r^2+b_2r^p)^{1/p} \leqq \|A\|/m \leqq (1-(cr\|A\|/m)^{p/s})^{1/p}$$

where $c=c(p)<1$.

---

[1] This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, with the United States Army Research Office, Contract No. DA-31-124-ARO-D-365.

In general, consider a Banach space $\mathfrak{B}$ with elements X and norm $\|X\|$ and the same definitions as before. Let $M=\max(\|X\|, \|Y\|)$.

DEFINITION I. $\mathfrak{B}$ is uniformly convex if, for each $\epsilon>0$, $\exists\delta>0$ such that each pair X, $Y\in\mathfrak{B}$ which satisfies $\|A\|\geqq(1-\delta)M$ also satisfies $\|D\|\leqq\epsilon M$.

DEFINITION II. $\mathfrak{B}$ is uniformly smooth if, for each $\sigma>0$, $\exists\tau>0$ such that each pair X, $Y\in\mathfrak{B}$ which satisfies $\|D\|\leqq\tau\|A\|$ also satisfies $\|X\|+\|Y\|\leqq 2\|A\|+\sigma\|D\|$.

THEOREM. *For* $1<p<\infty$, *L^p-space is* (a) *uniformly convex and* (b) *uniformly smooth.*

We first prove the theorem using the global inequalities and then prove the global inequalities (I) through local inequalities (II).

PROOF OF THEOREM. (a) We may take $\epsilon<1$ since by the triangle inequality $\|D\|<M$. By use of the inequalities $m<M$, (1), and $\|A\|\geqq(1-\delta)M$, we see that (Ia) implies that

$$1-\delta\leqq(1-(am^{-1}\|D\|)^{p/q})^{1/p}$$

or

$$a(m^{-1}\|D\|)^{p/q}\leqq 1-(1-\delta)^p = 1-(1-(a\epsilon)^{p/q}) = (a\,\epsilon)^{p/q}$$

for $\delta=1-(1-(a\epsilon)^{p/q})^{1/p}$ from which $\|D\|\leqq\epsilon m\leqq\epsilon M$.

(b) By inequality (Ib) and the fact that for $p>1$, $(1+|\xi|)^{1/p}<1+(1/p)|\xi|$,

$$m\leqq\|A\|(1+(1/p)(b_1r^2+b_2r^p)).$$

Thus if $\|D\|\leqq\tau\|A\|$,

$$m\leqq\|A\|+(1/p)(b_1\tau+b_2\tau^{p-1})\|D\| = \|A\|+(\sigma/2)\|D\|$$

if we choose $\tau(\sigma)$ as the positive solution of $\sigma=(2/p)(b_1\tau+b_2\tau^{p-1})$. Thus by (1), $\|X\|+\|Y\|\leqq 2\|A\|+\sigma\|D\|$.

The proof of (Ia) is based on a local inequality for vectors X, Y:

$$\text{(IIa)}\quad |D|^p\leqq\alpha(|X|^p+|Y|^p)^{1-s}(|X|^p+|Y|^p-2|A|^p)^s$$

with $\alpha>\alpha(p)$, $s=1$ for $p\geqq 2$, $s=p/2$ for $p<2$.

Using the Hölder inequality after integrating with respect to the measure μ we find since $0<q\leqq 1$,

$$\|D\|^p\leqq\alpha(\|X\|^p+\|Y\|^p)^{1-s}(\|X\|^p+\|Y\|^p-2\|A\|^p)^s.$$

Taking sth roots and rearranging we find (Ia) with $a=\alpha^{1/s}$.

PROOF OF THE LOCAL INEQUALITY (IIa). Clearly it suffices, by homogeneity, to prove (IIa) for vectors x, y satisfying $|x|=1$,

$|y| \leqq 1$. Let $a = \frac{1}{2}(x+y)$, $d = \frac{1}{2}(x-y)$. Then $1+|y|^p - 2|a|^p$ vanishes only for $d=0$ since

$$|a| < \tfrac{1}{2}|x| + \tfrac{1}{2}|y| \leqq \left(\frac{1 + |y|^p}{2}\right)^{1/p}$$

by convexity if $d \neq 0$. Hence, $\exists \beta > 0$ such that for all y, $|y| \leqq 1$,

$$1 + |y|^p - 2|a|^p > \beta |d|^2 \tag{2}$$

if there exists a neighborhood $\mathfrak{N}$ of x where such an inequality holds. We expand in the neighborhood of x, $|x| = 1$, by Taylor's series using $y = x - 2d$, $|y|^2 = 1 - 4d\cdot x + 4|d|^2$. Thus $|y|^p = 1 - 2pd\cdot x + 2p|d|^2 + 2p(p-2)(d\cdot x)^2 + O(|d|^3)$. From $|a|^p = (|x-d|^2)^{p/2}$ we have

$$|a|^p = 1 - pd\cdot x + \frac{p}{2}|d|^2 + \frac{p(p-2)}{2}(d\cdot x)^2 + O(|d|^3)$$

and thus

$$\begin{aligned} 1 + |y|^p - 2|a|^p &= p|d|^2(1 + (p-2)(d\cdot x/|d|)^2) + O(|d|^3) \\ &\geqq \beta |d|^2 \end{aligned}$$

for $|d|$ small enough; $\beta = 2(p-1)/p$ for $p<2$, $\beta = 2/p$ for $p \geqq 2$.

We apply (2) with $x = X/|X|$, $|X| \neq 0$, $y = Y/|X|$. After multiplying by $|X|^p$ we obtain

$$|X|^p + |Y|^p - 2|A|^p \geqq \beta |X|^{p-2}|D|^2. \tag{3}$$

For $p \geqq 2$ we note that $|D|^p \leqq |D|^2 |X|^{p-2}$ since $|X| \geqq |Y|$. Thus

$$|D|^p \leqq \beta^{-1}(|X|^p + |Y|^p - 2|A|^p)$$

so that the inequality (IIa) holds with $\alpha > \beta^{-1}$, $s=1$.

For $p<2$, we take the $p/2$ root of the inequality (3) obtaining

$$|D|^p \leqq \beta^{-p/2}|X|^{p(2-p)/2}(|X|^p + |Y|^p - 2|A|^p)^{p/2}$$

and the inequality (IIa) holds with $\alpha = \beta^{-s}$, $s = p/2$.

PROOF OF THE GLOBAL INEQUALITY (Ib). This is again proved by using a local inequality

$$\text{(IIb)} \quad \tfrac{1}{2}(|X|^p + |Y|^p) \leqq |A|^p + b_1 |A|^{p-2}|D|^2 + b_2|D|^p$$

where $b_1 = b_1(p)$ vanishes for $p \leqq 2$, $b_2 = b_2(p)$.

Using Hölder's inequality we obtain

$$\tfrac{1}{2}(\|X\|^p + \|Y\|^p) \leqq \|A\|^p + b_1\|A\|^{p-2}\|D\|^2 + b_2\|D\|^p$$

since $b_1=0$ for $p\leqq 2$.

PROOF OF LOCAL INEQUALITY (IIb). Consider the vectors x, y with $a=(\tfrac{1}{2}(x+y), d=\tfrac{1}{2}(x-y)$ and which satisfy $|a|=1$. Then $|x|^p=|a+d|^p=(1+2a\cdot d+|d|^2)^{p/2}$. Clearly for $|d|<d_1$ say, $|x|^p=1+pa\cdot d+O(|d|^2)\leqq 1+pa\cdot d+b_1|d|^{2s}$ where s is 1 for $p>2$, and $p/2$ for $p<2$ and $b_1=b_1(d_1, p)$. But for $|d|>d_1$

$$|x|^p < (|d|^2/d_1^2 + 2|d|^2/d_1 + |d|^2)^{p/2} < b_2^*|d|^p.$$

Combining these two inequalities for some fixed d_1 we can find $b_1=b_1(p)$, $b_2=b_2(p)$ such that

$$|x|^p \leqq 1 + pa\cdot d + b_1|d|^2 + b_2|d|^p$$

where $b_1=0$ for $p<2$.

Similarly

$$|y|^p = |a-d|^p = 1 - pa\cdot d + b_1|d|^2 + b_2|d|^p.$$

Adding we find

$$\tfrac{1}{2}|x|^p + \tfrac{1}{2}|y|^p \leqq 1 + b_1|d|^2 + b_2|d|^p \tag{4}$$

for $|x+y|=1$, $b_1=0$ for $p<2$. Assuming $|A|\neq 0$ we apply (4) to $x=X/|A|$, $y=Y/|A|$ and multiply by $|A|^p$. Thus

$$\tfrac{1}{2}(|X|^p + |Y|^p) \leqq |A|^p + b_1|A|^{p-2}|D|^2 + b_2|D|^p.$$

If $A=0$, $X=-Y$, and $\tfrac{1}{2}(|X|^p+|Y|^p)=2^{p-1}|D|^p$ so that the local inequality holds by modifying b_2.

REFERENCES

1. K. O. Friedrichs, *Formally dual spaces, uniform rotundity, and representation theorems, A chapter from a course on functional analysis*, New York University, New York, 1968.

2. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

3. G. D. Mostov, *Quasi-conformal mappings in n-space*, Inst. Hautes Études Sci. Publ. Math. no. 14, 1969, pp. 53–104.

4. M. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1958.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK 10012