A NOTE ON THE NUMBER OF INTEGRAL IDEALS OF BOUNDED NORM IN A QUADRATIC NUMBER FIELD

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Let K be an algebraic number field of degree 2 and $F(n)$ the number of nonzero integral ideals of norm n in K. Define $P(x)$ by

$$\sum_{n \leq x} F(n) = \lambda h x + P(x),$$

where h denotes the class number of K and

$$\lambda = 2^{r_1+r_2}\pi^{r_2}R/(w\sqrt{|\Delta|}),$$

where r_1 is the number of real conjugates, $2r_2$ the number of imaginary conjugates, R the regulator, w the number of roots of unity, and Δ the discriminant of K. It is known that [8, Satz 210] $P(x) = O(x^{1/3})$. On the other hand, Landau [9] also showed that

$$P(x) = \Omega_{\pm}(x^{1/4}).$$

Improvements were made by Szegö and Walfisz [10] and Chandrasekharan and Narasimhan [2], [3]. The former authors showed that if K is imaginary,

$$P(x) = \Omega_{-}(\{x \log x\}^{1/4}),$$

and if K is real

$$P(x) = \Omega_{+}(\{x \log x\}^{1/4}).$$

The latter showed that

$$\limsup_{x\to\infty} \inf P(x)/x^{1/4} = \pm \infty. \tag{1}$$

In 1961 Gangadharan [5], improving a method of Ingham, made improvements on (1) for the corresponding problems on $r(n)$, the number of representations of n as the sum of two squares, and $d(n)$, the number of divisors of n. Using Gangadharan's method, we can obtain improvements on (1) for our problem. Before stating this result we must make some definitions.

DEFINITION 1. Let $S_x (x \geqq 2)$ be the set of all real numbers η expressible in the form

$$\eta = \left| \sqrt{n} + \sum_{k=1}^{N} r_k \sqrt{q_k} \right|,$$

where $q_1, \ldots, q_N$ are the square-free integers less than or equal to x, and n and r_k, $k=1, \cdots, N$, are integers such that

$$n \geqq 0 \qquad |r_k| \leqq 1, \qquad \sum_{k=1}^{N} |r_k|^2 \geqq 2.$$

It follows that [5, pp. 700–701] that there is a unique $\tilde{\eta} \in S_x$ such that $0 < \tilde{\eta}(x) < 1$ and if $\eta \in S_x$ then $\eta \geqq \tilde{\eta}$.

DEFINITION 2. Let $q(x) = -\log \tilde{\eta}(x)$. Define C_q to be the class of all functions $Q(x)$ such that for $x \geqq x_Q$, $Q(x)$ is continuous, $Q(x)/x$ increases with x, and $Q(x) \geqq q(x)$.

It follows that for $x \geqq x_Q'$, $Q^{-1}(x)$ exists, is continuous and increasing, and tends to ∞ with x.

THEOREM. *If $Q(x) \in C_q$, then, as x tends to ∞,*

$$P(x) = \Omega_{\pm}(\{xQ^{-1}(\log x)\}^{1/4}). \tag{2}$$

It can be shown that [5, pp. 701–703] for some constant $b > 2$. $b^{x/\log x} \in C_q$. We have then the following

COROLLARY. *As x tends to ∞,*

$$P(x) = \Omega_{\pm}(\{x(\log \log x)(\log \log \log x)\}^{1/4}).$$

The proof of (2) depends upon two identities. Let Re $s > 0$. If K is imaginary and $B = 2\pi/\sqrt{|\Delta|}$,

$$\sum_{n=1}^{\infty} F(n) e^{-s\sqrt{n}} = \frac{2\lambda h}{s^2} - \frac{\lambda h}{B} + 2Bs \sum_{n=1}^{\infty} \frac{F(n)}{(s^2 + 4B^2 n)^{3/2}}. \tag{3}$$

If K is real and $B = \pi/\sqrt{|\Delta|}$,

$$\begin{aligned} &\sum_{n=1}^{\infty} F(n) e^{-s\sqrt{n}} \\ &= \frac{2\lambda h}{s^2} + \frac{1}{2B\pi} \sum_{n=1}^{\infty} \frac{F(n)}{n} \cdot \left\{ l\left(\frac{-s}{2B\sqrt{n}}\right) \right. \\ &\qquad \left. - \tfrac{1}{2} l\left(\frac{-is}{2B\sqrt{n}}\right) - \tfrac{1}{2} l\left(\frac{is}{2B\sqrt{n}}\right) \right\}, \end{aligned} \tag{4}$$

where

$$l(s) = \int_0^{\pi/2} \frac{\sin \phi d\phi}{(1 - \frac{1}{2}s \sin \phi)^2}.$$

A proof of (3) can be found in [1], although an easier proof can be given along the same lines as that of Hardy [6] for a similar identity involving $r(n)$. The proof of (4) is more complicated, but, using primarily the functional equation for the associated zeta-function, one can establish (4) by a method of Hardy [7] for a similar identity involving $d(n)$.

The proof of (2) now follows along the same lines as that in [5]. We remark that in the proof one needs the following two facts. We have $F(n) = O(n^\epsilon)$ for every $\epsilon > 0$; this holds for any algebraic number field [4]. When K is a quadratic field, it is not difficult to show that $F(m^2n) \leqq F(m^2)F(n)$.

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