APPROXIMATING HOMOTOPIES BY ISOTOPIES IN FRÉCHET MANIFOLDS

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Let M be an F-manifold, that is, a separable, metric manifold modelled on an infinite-dimensional Fréchet space. The question was raised at a problem seminar this January (1969) at Cornell University whether homotopic embeddings of another F-manifold in M are isotopic. In this note the affirmative answer is given and a stronger result established.

Given an open cover $\mathcal{U}$ of a space X, two maps f and g of a space Y into X are said to be *$\mathcal{U}$-close* provided that for each y in Y there is an element of $\mathcal{U}$ containing both $f(y)$ and $g(y)$. The two maps are said to be *pseudo-isotopic* provided there is a map $h: Y \times I \to X$ with

$$h(y, 0) = f(y), \qquad h(y, 1) = g(y)$$

and which for each t in $(0, 1)$ is an embedding of $Y \times \{t\}$. The theorem is as follows:

THEOREM. *Homotopic maps of a separable metric space into an F-manifold are pseudo-isotopic. If the domain is complete, the pseudo-isotopy may be required to be through closed embeddings. Furthermore, given any open cover $\mathcal{U}$ of the manifold and any homotopy F between the maps, the pseudo-isotopy may be required to be $\mathcal{U}$-close to F.*

PROOF. Let X be the separable metric space, M the F-manifold, and f and g the homotopic maps of X into M. By a collection of results, all separable, infinite-dimensional Fréchet spaces are homeomorphic to the countably infinite product s of open intervals $(-1, 1)$. (For a discussion of these results and a bibliography, see the introduction of [3].) Furthermore, a theorem of R. D. Anderson and R. M. Schori [4] asserts that given any open cover $\mathcal{U}$ of M, there is a homeomorphism $h_{\mathcal{U}}$ of M onto $M \times s$ so that $p \circ h_{\mathcal{U}}$ is $\mathcal{U}$-close to the identity map, where p is the projection onto M. If $\{s_i\}_{i=1}^{\infty}$ is a countable, indexed family of copies of s, it is easy to see that s', the product of the s_i's, is homeomorphic to s, so s may be replaced by s' in the above theorem.

For each integer i and real number t in $(-1, 1)$, let $\psi_{i,t}: s_i \to s_i$ be the map which multiplies in each coordinate by t, and let

$$\begin{aligned}
\phi(i, t) &= 1 && \text{if } t \leqq \frac{1}{i+1} \text{ or } t \geqq \frac{i}{i+1}, \\
&= 0 && \text{if } \frac{1}{i} \leqq t \leqq \frac{i-1}{i}, \\
&= (i+1)(1-it) && \text{if } \frac{1}{i+1} \leqq t \leqq \frac{1}{i}, \\
&= (i+1)(it-i+1) && \text{if } \frac{i-1}{i} \leqq t \leqq \frac{i}{i+1}.
\end{aligned}$$

Also, let k_i be an embedding of X in s_i, as a closed set if X is complete. (It is well known that this may be done in a separable Banach space.)

Given any homotopy F between f and g and any open cover $\mathfrak{U}$ of M, let $\mathfrak{V}$ be a star-refinement of $\mathfrak{U}$; $h_{\mathfrak{V}}$, a homeomorphism of M onto $M\times s'$ such that $p \circ h_{\mathfrak{V}}$ and the identity are $\mathfrak{V}$-close, and define G: $X\times I\to M$ by

$$G(x, t) = h_{\mathfrak{V}}^{-1} \circ \left[\mathrm{id}_M \times \prod_{i=1}^{\infty} (\psi_{i,\phi(i+2,t)} + (\psi_{i,1-\phi(i+1,t)} \circ k_i(x))) \right] \circ h_{\mathfrak{V}} \circ F(x, t),$$

where "$+$" is understood to indicate coordinate-wise addition, and "$\prod$", the product of mappings.

For each t in $(0, 1)$, $h_{\mathfrak{V}} \circ G | X\times\{t\}$ may be regarded as the product of a mapping of X into $M\times\prod_{i=1,i\neq i_0}^{\infty} s_i$ with a (closed) embedding of X in s_{i_0}, where i_0 is any integer greater than or equal to both $1/t$ and $1/(1-t)$. It is a simple matter to see that this is a (closed) embedding since it is continuous, one-to-one, and the inverse is continuous because given a point (x, t) and a sequence $\{(x_i, t)\}_{i=1}^{\infty}$ in $X\times\{t\}$ for which $h_{\mathfrak{V}} \circ G(x_i, t)$ converges to $h_{\mathfrak{V}} \circ G(x, t)$, the s_{i_0}-coordinates of $\{h_{\mathfrak{V}} \circ G(x_i, t)\}_{i=1}^{\infty}$ converge to the s_{i_0}-coordinate of $h \circ G(x, t)$, and as the mapping into the s_{i_0}-coordinate is an embedding, this forces $\{x_i\}_{i=1}^{\infty}$ to converge to x. The image $h_{\mathfrak{V}} \circ G(X\times\{t\})$ is closed if X is complete, since if $\{(x_i, t)\}_{i=1}^{\infty}$ is a sequence in $X\times\{t\}$ and p is in $M\times s'$ with $h_{\mathfrak{V}} \circ G(x_i, t)$ converging to p, then the s_{i_0}-coordinates of $h_{\mathfrak{V}} \circ G(x_i, t)$ converge to the s_{i_0}-coordinate of p, which forces the s_{i_0}-coordinate of p to be $k_{i_0}(x)$, for some x, and thus forces $\{x_i\}_{i=1}^{\infty}$ to converge to x.

If it is desired, the pseudo-isotopy may be modified slightly to provide that

(a) it be an embedding of $X\times(0, 1)$ in M and (b) the image of $X\times(0, 1)$ lie in a countable union of closed sets of M each of which has Property Z in M (in case X is complete, the image of $X\times(0, 1)$ may be required to be the countable union of closed sets with Property Z in M).

(A closed set Y has Property Z in M provided that for each nonnull open set U of M with trivial homotopy groups, $U-Y$ be also nonnull and have trivial homotopy groups. The importance of Property Z for F-manifolds is demonstrated by [2] in which it is shown that the subsets of such which are homeomorphic to the manifolds by homeomorphisms $\mathfrak{U}$-close to the identity for all open covers $\mathfrak{U}$ are precisely the complements of countable unions of closed sets, each with Property Z.) The modified homotopy $H: X\times I\to M$ may be defined by setting $H(x, t)=h_{\mathfrak{V}}^{-1}\circ[id_M\times\prod_{i=1}^{\infty}\xi_{i,t}(x)]\circ h_{\mathfrak{V}}\circ F(x, t)$, where

$$\begin{aligned}\xi_{i,t}(x) &= \psi_{i,\phi(i+2,t)} + (\psi_{i,1-\phi(i+1,t)}\circ k_i(x)), \quad \text{if } i \text{ is even,}\\ &= \psi_{i,\phi(i+2,t)} + (\psi_{i,1-\phi(i+1,t)}\circ \psi_{i,t}(y_i)), \quad \text{if } i \text{ is odd}\\ &\qquad\qquad \text{but not divisible by three, and}\\ &= \psi_{i,\phi(i+2,t)}, \quad \text{if } i \text{ is an odd multiple of three.}\end{aligned}$$

Here, y_i is merely a point in s_i with not all coordinates zero; the y_i's are introduced for the purpose of guaranteeing that $H|X\times(0, 1)$ be an embedding. The insertion of merely the $\psi_{i,\phi(i+2,t)}$ in infinitely many coordinates is to ensure that for any t_0 in $(0, 1/2)$, $h_{\mathfrak{V}}\circ H(X)\times[t_0, 1-t_0])$ project into s' on a set of infinite co-dimension which, by a theorem of R. D. Anderson [1], must have closure with Property Z. This guarantees that the closure of $h_{\mathfrak{V}}\circ H(X\times[t_0,1-t_0])$ has Property Z in $M\times s'$ and hence that the closure of $H(X\times[t_0, 1-t_0])$ has Property Z in M. If X is complete, the construction gives that $H(X\times[t_0, 1-t_0])$ is closed and has Property Z for each t_0 in $(0, 1/2)$.

REMARK. D. W. Henderson has recently proven in [5] that if X is an F-manifold and $\mathfrak{U}$ an open cover of M, than any map of X into M may be approximated $\mathfrak{U}$-closely by closed and open embeddings.

In light of these results, the following question, also raised at Cornell, would seem to be the appropriate one: "Under which circumstances are two homotopic embeddings of one F-manifold in another ambient isotopic?"

REFERENCES

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