

**FUNCTION THEORETIC METHODS IN THE
THEORY OF BOUNDARY VALUE PROBLEMS
FOR GENERALIZED METAHARMONIC
FUNCTIONS¹**

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1. Introduction. Although existence and uniqueness theorems for linear elliptic partial differential equations in a domain D with coefficients continuous in D have been known for some time, similar results for equations whose coefficients have singularities in the domain under consideration are practically unknown. Recently attention has been given to a class of singular equations which appear frequently in both pure and applied mathematics and are known as generalized axially symmetric partial differential equations [3], [5], [8]. Just as a thorough knowledge of the Laplace and Helmholtz equation guided the attack on linear elliptic equations with continuous coefficients, it is hoped that a better understanding of generalized axially symmetric equations will give insight towards developing a theory of elliptic equations with singular coefficients.

We wish to announce in this note a uniqueness theorem for the exterior Dirichlet problem for the generalized axially symmetric metaharmonic equation

$$(1) \quad L_{\lambda, s}[u] \equiv \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial \rho^2} + \frac{s}{\rho} \frac{\partial u}{\partial \rho} + \lambda^2 u = 0$$

where $s > -1$, $s \neq 0$, and $\lambda > 0$ [5]. This is the first time a uniqueness theorem has been obtained for a singular elliptic partial differential equation in more than two variables whose coefficients are singular in its domain of definition, such work in the present case having been delayed due to an insufficient knowledge of certain areas of the theory of several complex variables. Our result depends on first using the Hadamard-Gilbert Theorem [5] to solve the classical expansion problem for Appell series and to then apply this along with Vitali's theorem for several complex variables [6] to obtain the desired uniqueness theorem.

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2. The expansion problem for Appell series. An Appell series is a series of the form

$$(2) \quad \sum_M a_M V_M^{(s)}(\xi) \equiv \sum_{\mu=0}^{\infty} \sum_{m=\mu} a_M V_M^{(s)}(\xi); \quad s > -1, s \neq 0$$

where the polynomials

$$(3) \quad V_M^{(s)}(\xi) \equiv V_M^{(s)}(\xi_1, \dots, \xi_n); \quad M = (m_1, m_2, \dots, m_n)$$

are uniquely defined by the generating function

$$(4) \quad (1 - 2(\alpha, \xi) + \|\alpha\|^2)^{-(n+s-1)/2} = \sum \alpha^M V_M^{(s)}(\xi).$$

Here $(\alpha, \xi) \equiv \sum_{i=1}^n \alpha_i \xi_i$, $\|\alpha\|^2 = (\alpha, \alpha)$, $\alpha^M \equiv \alpha_1^{m_1} \dots \alpha_n^{m_n}$, $m = |M| = m_1 + m_2 + \dots + m_n$ and the summation in equation (2), (4), and in what follows, is meant to be an n -fold sum over all indices from zero to infinity. We also need the related polynomials

$$(5) \quad U_M^{(s)}(\xi) \equiv U_M^{(s)}(\xi_1, \xi_2, \dots, \xi_n)$$

which are defined by the generating function

$$(6) \quad \{[(\alpha, \xi) - 1]^2 + \|\alpha\|^2(1 - \|\xi\|^2)\}^{-s/2} = \sum \alpha^M U_M^{(s)}(\xi).$$

These polynomials satisfy the biorthogonality relation

$$(7) \quad \int_{S(0,1)} (1 - \|\xi\|^2)^{(s-1)/2} V_M^{(s)}(\xi) U_L^{(s)}(\xi) d^n \xi = \delta_{LM} \frac{2\pi^{n/2} \Gamma(s/2 + 1)(s)m}{(2m + n + s - 1)\Gamma(n/2 + s/2 - 1/2)M!},$$

$\delta_{LM} \equiv \delta_{i_1 m_1} \dots \delta_{i_n m_n}$, $M! = m_1! \dots m_n!$. Here $S(0, 1)$ is the real solid n dimensional ball $\{\xi \mid \|\xi\| \leq 1\}$. For more information concerning the polynomials $V_M^{(s)}(\xi)$ and $U_M^{(s)}(\xi)$ see [1]. We are interested in the classical expansion problem for analytic functions of n complex variables in terms of the polynomials $V_M^{(s)}(\xi)$ viz. if $f(\xi)$ is analytic on $S(0, 1)$ can it be expanded in a series of the form (2) and if so what can be said about the region of convergence?

THEOREM 1. *Let $f(\xi)$ be an analytic function of n complex variables in some neighborhood $\bar{\eta}$ of the unit ball $S(0, 1)$. Then $f(\xi)$ can be expanded in an Appell series,*

$$f(\xi) = \sum a_M V_M^{(s)}(\xi); \quad s > -1, s \neq 0$$

which converges uniformly for $\xi \in \mathfrak{S} \equiv \eta \{ S(0, 1) \} \cap \eta^* \{ S(0, 1) \}$ with $\eta \{ S(0, 1) \} \subset \bar{\eta}$, $\eta^* \equiv \{ \xi | \xi^* \in \eta \}$ (* denotes complex conjugation), and where the coefficients are given by the formula

$$a_M = h_M^* \int_{S(0,1)} (1 - \|\xi\|)^{(s-1)/2} f(\xi) U_M^{(s)}(\xi) d^n \xi$$

with

$$h_M^* = \frac{(2m + n + s - 1) \Gamma(\frac{1}{2}[n + s - 1]) M!}{2\pi^{n/2} \Gamma(s/2 + 1) (s)_m}$$

OUTLINE OF PROOF. The technique used is to develop an integral operator approach to the method of generating kernels [2]. We first define a new function $F(\zeta)$ defined by

$$(8) \quad F(\zeta) = Of(\xi) = \left(\frac{1}{2\pi i}\right)^n \int_{S(0,1)} K_1(\zeta, \xi) f(\xi) d^n \xi = \sum a_M \zeta^M$$

where

$$(9) \quad K_1(\zeta, \xi) \equiv \frac{\Gamma(\frac{1}{2}[n + s - 1]) (1 - \|\xi\|^2)^{(s-1)/2}}{2\pi^{n/2} \Gamma(s/2 + 1)} \cdot \sum \frac{2m + n + s - 1}{(s)_m} M! U_M^{(s)}(\xi) \zeta^M$$

From the Weierstrass comparison theorem [5] and the generating function expansion (6) it can be shown that equation (8) defines an analytic function of ζ for $\zeta \in \Delta(0, 1)$. Here $\Delta(0, 1)$ denotes the open unit ball in the complex ζ space. We next define the inverse operator O^{-1} by

$$(10) \quad f(\xi) = O^{-1}F(\zeta) = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma} K_2(\xi, \zeta) F(\zeta) \frac{d^n \zeta}{\zeta} = \sum a_M V_M^{(s)}(\xi)$$

where

$$K_2(\xi, \zeta) \equiv \sum \zeta^{-M} V_M^{(s)}(\xi) = \left(1 - 2\left(\frac{1}{\zeta}, \xi\right) + \left\|\frac{1}{\zeta}\right\|^2\right)^{-(n+s-1)/2}$$

and $\Gamma = \prod_{i=1}^n \Gamma_i$ where the Γ_i are chosen such that Γ lies outside a sufficiently large hypersphere $\Delta(0, R)$. By using the Hadamard-Gilbert theorem [5] and the fact that $f(\xi)$ is analytic in $\bar{\eta}$, one can

conclude from (8) that $F(\zeta)$ must in fact be an analytic function in some complex neighborhood of $\overline{\Delta(0, 1)}$ on its principal sheet of definition. Using, this fact in conjunction with the series representation (8) for $F(\zeta)$ shows that the series (10) converges uniformly in some complex neighborhood of $S(0, 1)$ and agrees with $f(\xi)$ there. The formula for the coefficients is arrived at through use of the biorthogonality relation (7).

3. The uniqueness theorem. Since the plane $\rho=0$ is a singular curve of the regular type with indices 0 and $1-s$ [4] there always exist solutions of equation (1) which are real analytic on some portion of the axis $\rho=0$, and if $s > -1$, $s \neq 0$, such solutions can be continued across the axis as an even function of ρ . Hence each such analytic solution is analytic in a domain D that is symmetric with respect to the axis $\rho=0$ and can therefore be expressed as $u(x, \rho) = \bar{u}(r, \xi)$ where $r, \xi = (\xi_1, \dots, \xi_n)$ are hyper-zonal coordinates.

THEOREM 2. Assume $s > -1$, $s \neq 0$, and let $u(x, \rho) = \bar{u}(r, \xi)$ be a real analytic solution of $L_{\lambda_s}[u] = 0$ in \mathfrak{F} where \mathfrak{F} is the exterior of a normal domain \mathfrak{D} which is symmetric with respect to the axis $\rho=0$. Let $\bar{u}(r, \xi) \in C^2(\mathfrak{F})$ and for each fixed r , $a \leq r < \infty$ (where a is such that $r=a$ contains \mathfrak{D}) assume that $\bar{u}(r, \xi)$ is an analytic function of ξ in a domain \mathfrak{S} in the complex ξ space containing the (real) closed unit ball $S(0, 1)$, where \mathfrak{S} is symmetric with respect to conjugation and independent of r . If

(i) there exists a positive constant M such that

$$r^{(n+s)/2} |\bar{u}(r, \xi)| \leq M \quad \text{for } a \leq r < \infty, \quad \xi \in \mathfrak{S},$$

(ii) $\lim_{r \rightarrow \infty} \int_{S(0,1)} r^{n+s} (1 - \|\xi\|^2)^{(s-1)/2} |\partial \bar{u} / \partial r - i \lambda \bar{u}|^2 dS = 0$,

(iii) $\bar{u}(r, \xi) = 0$ on the boundary of \mathfrak{D}

then $\bar{u}(r, \xi) \equiv 0$.

REMARK. If conditions (i) and (ii) of the theorem are not imposed there will exist eigenfunctions of equation (1) and we cannot expect a unique solution to exist.

OUTLINE OF PROOF. Using Theorem 1 we expand $\bar{u}(r, \xi)$ for each fixed $r \geq a$ in an Appell series

$$(11) \quad \bar{u}(r, \xi) = \sum a_M(r) V_M^{(s)}(\xi); \quad \xi \in \mathfrak{S}.$$

From the biorthogonality property (7) it can be shown that

$$(12) \quad a_M(r) = r^{-1/2(n+s-1)} Z_{\mu+1/2(n+s-1)}(\lambda r)$$

where $Z_\nu(\lambda r)$ denotes a cylinder function of order ν . Applying Green's formula and condition (ii) gives

$$(13) \quad \lim_{r \rightarrow \infty} \int_{S(0,1)} r^{n+s} (1 - \|\xi\|^2)^{(s-1)/2} |u|^2 d^n \xi = 0.$$

If the polynomials $V_M^{(g)}(\xi)$ satisfied Parseval's relation we could now proceed with the proof by following the approach used by Vekua in [7]. Since this is not the case we resort to techniques first used by Colton in [3]. Define

$$(14) \quad g(r, \xi) \equiv \bar{u}(r, \xi) \bar{u}^*(r, \xi^*),$$

and observe that $g(r, \xi)$ is an analytic function of ξ which agrees with $|u(r, \xi)|^2$ for ξ real valued. From condition (i) of the theorem it can be shown that $r^{n+s} |g(r, \xi)|$ is uniformly bounded for $a \leq r < \infty, \xi \in \mathfrak{S}$. Now define for fixed r and variable ξ

$$(15) \quad F_r(\xi) \equiv \int_0^\xi r^{n+s} (1 - \|\xi\|^2)^{(s-1)/2} g(r, \xi) d^n \xi,$$

and note that $F_r(\xi)$ is analytic in $\Delta(0, \epsilon)$ (if ϵ is chosen sufficiently small) and for $r \geq a, \xi \in \Delta(0, \epsilon), |F_r(\xi)|$ is uniformly bounded. By applying a version of Vitali's theorem for several complex variables [6], it can be shown using equations (13) and (15) that

$$(16) \quad \lim_{r \rightarrow \infty} r^{(n+s)/2} \bar{u}(r, \xi) = 0; \quad \xi \in S(0, 1).$$

Theorem 1, equation (12), and the asymptotic behavior of cylinder functions (c.f. [7]) now shows that $a_M(r) = 0$ for every M and hence $\bar{u}(r, \xi) \equiv 0$.

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