# A LOCAL SPECTRAL THEORY FOR OPERATORS. II

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Communicated by Henry Helson, November 22, 1968

I. **Introduction.** Let $T$ be an operator on a Banach space $B$. Let $\sigma(T)$, the spectrum of $T$, lie on a line, a circle, or, more generally, a smooth curve. If the resolvent $R_z(T)=(T-zI)^{-1}$ satisfies a growth condition with respect to $\sigma(T)$, it is possible, in many cases, to develop an invariant subspace decomposition for $T$. We mention explicitly the work of Bartle [**1**], Godement [**4**], Leaf [**6**], Lorch [**7**], Schwartz [**13**], Wermer [**19**], and Wolf [**20**]. Since, in the references cited, none of the subspaces is necessarily complemented by an invariant subspace, one can not expect this invariant subspace decomposition to generate a countably additive resolution of the identity. Such a spectral resolution is precisely the achievement of the Dunford theory [**3**], but there it was necessary to assume a second condition in order to obtain it. This condition (Dunford Boundedness) is not easy to verify in practice.

In this note, we will study several situations in Hilbert space, where a strong growth condition on the resolvent is sufficient to guarantee a countably additive resolution of the identity, i.e., the operator turns out to be similar to a normal operator. The results in §3 generalize, and are dependent on, some recent work of Gokhberg and Krein. We will only sketch proofs. Complete details will appear in [**16**] and elsewhere.

From now on, the underlying space is always a Hilbert space. All operators are bounded. By a smooth Jordan curve, we mean a Jordan curve of class $C^2$ (in the complex plane).

II. In this section we study conditions on the resolvent which insure normality.

LEMMA 1. *Let* $\|(T-\lambda)^{-1}\|\leqq 1/d$ *where* $0<d<|\lambda|$. *Then*

$$\left\|\left(T^{-1}-\frac{\bar{\lambda}}{|\lambda|^2-d^2}\right)^{-1}\right\|\leqq\frac{|\lambda|^2-d^2}{d}.$$

THEOREM 1. *Let $U$ be an open set and let $\sigma(T)\cap U$ lie in the smooth Jordan curve $C$. Let $\|R_\lambda(T)\|\leqq 1/\text{dist }[\lambda, C]$ for $\lambda\in U$. Then $T=T_1\oplus T_2$ where $T_1$ is normal, $\sigma(T_1)=\text{closure }[\sigma(T)\cap U]$ and $\sigma(T_2)\subset\sigma(T)\cap U'$.*

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[1] The author gratefully acknowledges the support of the National Science Foundation.

*($\oplus$ denotes orthogonal direct sum.)*

The proof of Theorem 1 is too long to even sketch. It relies heavily on successive application of Lemma 1 coupled with material on the numerical range found in [**15**].

COROLLARY 1. *Let $\sigma(T)$ lie in the smooth Jordan curve $C$. Let $\|R_\lambda(T)\| \leqq 1/\text{dist}[\lambda, C]$ for $\lambda$ in a neighborhood of $C$. Then $T$ is normal.*

Corollary 1 was proved for $C$ = Reals by Nieminen [**10**] and $C$ = unit circle by Donoghue [**2**] with a slightly stronger growth condition.

Since $\|R_\lambda(T)\| = 1/\text{dist}[\lambda, \sigma(T)]$ for any hyponormal operator $T$, Theorem 1 is useful in dealing with this class of operators. (See for example the question raised by Putnam in [**12**]; also Putnam [**11**] and Stampfli [**14**].)

DEFINITION. An operator $T$ on a Hilbert space $H$ is in $\mathfrak{C}_\rho$ if there exists a Hilbert space $K \supset H$, a constant $\rho > 0$, and a unitary operator $U$ on $K$, such that $T^n = \rho P U^n P$ for $n = 1, 2, \cdots$, where $P$ is the self-adjoint projection of $K$ on $H$.

COROLLARY 2. *If $T \in \mathfrak{C}_\alpha$ and $T^{-1} \in \mathfrak{C}_\beta$, then $T$ is unitary.*

III. In this section, we will generalize some recent work of Gokhberg and Krein [**5**]. First, we state their theorem which depends on a deep result of Nagy and Foiaş [**17**].

THEOREM (G–K). *Let $T$ be a contraction. If $\|R_\lambda(T)\| \leqq K/(1 - |\lambda|)$ for $|\lambda| < 1$, then $T$ is similar to a unitary operator.*

The next lemma is a modest improvement.

LEMMA 2. *Let*
(i) $\|R_\lambda(T)\| \leqq 1/(|\lambda| - 1)$ *for* $|\lambda| > 1$, *and*
(ii) $\|R_\lambda(T)\| \leqq K/(1 - |\lambda|)$ *for* $|\lambda| < 1$.
*Then $T$ is similar to a unitary.*

PROOF. Condition (i) implies that $W(T)$, the numerical range of $T$, lies in the unit disc (and conversely). Hence, by a result of Nagy and Foiaş [**18**], $T = QAQ^{-1}$ where $A$ is a contraction. But $\|R_\lambda(A)\| \leqq \|Q\| \, \|Q^{-1}\| K/(1 - |\lambda|)$ for $|\lambda| < 1$. Thus, $A$ and hence $T$ are similar to a unitary operator by the previous theorem.

THEOREM 2. *Let $\sigma(T)$ lie in the unit circle. Let*
(i) $\|R(T)\| \leqq K/(1 - |\lambda|)$ *for* $\alpha < |\lambda| < 1$, *and*
(ii) $\|R(T)\| \leqq 1/(|\lambda| - 1)$ *for* $1 < |\lambda| < \beta$.
*Then $T$ is similar to a unitary operator.*

PROOF. By suitable use of Lemma 1, we can reduce the proof to the case where $T$ satisfies (i) and (ii) in a sector (from 0 to infinity). Let $\delta$ be a small arc of the unit circle, contained in this sector. There exists an invariant subspace $H_\delta$ of $T$ such that $\sigma(T|H_\delta)\subset\delta$. Moreover, it follows from the Lorch approximation theorem [7], and the growth conditions that $W(T|H_\delta)$ is contained in the unit disk. Hence, $T|H_\delta$ is similar to a unitary by Lemma 2. Unfortunately, it is not clear that there exits a subspace complementary to $H_\delta$ which is invariant under $T$. This difficulty can be overcome by cutting $T$ down to a subset of $\delta$ and estimating the angle between appropriate subspaces. Repeating this argument a finite number of times completes the proof.

REMARK. It makes no difference if the roles of $K$ and 1 are interchanged in (i) and (ii). Moreover, (i) and (ii) are not needed for the entire circle, but only in a neighborhood of $\sigma(T)$. In fact, one can even recover a variation of Theorem 1. Let $U$ be an open set and let $U\cap\sigma(T)\neq\emptyset$ lie in the unit circle. Further, let $R_\lambda(T)$ satisfy (i) and (ii) for $\lambda\in U$. Then, $T=T_1\dotplus T_2$, where $T_1$ is similar to a normal and $\sigma(T_1)$ can be chosen to be any closed subset of $\sigma(T)$ contained in $U$. ($\dotplus$ denotes direct sum, i.e., the underlying spaces are complementary.)

Theorem 1 can be used to obtain results on operators with real spectrum.

COROLLARY 1. *Let $\sigma(T)$ be real. Let*
(i) $\|R_\lambda(T)\| \leqq 1/\operatorname{Im}\lambda$ *for* $0<\operatorname{Im}\lambda<\alpha$, *and*
(ii) $\|R_\lambda(T)\| \leqq K/|\operatorname{Im}\lambda|$ *for* $\beta<\operatorname{Im}\lambda<0$.
*Then $T$ is similar to a selfadjoint operator.*

Since growth conditions on the resolvent are usually applied near the spectrum or at infinity, the next corollary, at first glance, seems surprising.

COROLLARY 2. *Let $\sigma(T)$ be real. Let $\|R_\lambda(T)\| \leqq 1/\operatorname{Im}\lambda$ for $\lambda$ in a neighborhood of the parabola $2y=x^2+1\ (z=x+iy)$. If $\|R_\lambda(T)\|\leqq 1/|\operatorname{Im}\lambda|$ for $\lambda$ in a neighborhood of $2y=-(x^2+1)$ then $T$ is selfadjoint. If $\|R_\lambda(T)\|\leqq K/|\operatorname{Im}\lambda|$ for $\beta<\operatorname{Im}\lambda<0$, then $T$ is similar to a selfadjoint operator.*

Corollaries 1 and 2 follow easily by taking the Cayley transform of $T$. The remark following Theorem 2 applies here as well.

IV. The growth condition on the resolvent in the preceding section can not be substantially weakened as seen by the following

EXAMPLE (MCCARTHY AND SCHWARTZ [9], A. S. MARKUS [8]). There exists an operator $T$ on Hilbert space, such that the spectrum of $T$ is real, and $\|R_\lambda(T)\| \leqq K/|\mathrm{Im}\lambda|$ for all $\lambda$. However, $T$ is not similar to a selfadjoint operator.

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