DIFFERENTIABILITY THEOREMS FOR WEAK SOLUTIONS OF NONLINEAR ELLIPTIC DIFFERENTIAL EQUATIONS[1]

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I shall begin by speaking about the extremals of an integral of the form

$$I(z, G) = \int_G f[x, z(x), \nabla z(x)]dx \tag{1}$$

where G is a domain in R^ν,

$$x = (x^1, \cdots, x^\nu), \qquad z = (z^1, \cdots, z^N), \qquad dx = dx^1 \cdots dx^\nu, \tag{2}$$

$z(x)$ is a vector function, ∇z denotes its gradient which is the set of functions $\{z^i_{,\alpha}\}$ where $z^i_{,\alpha}$ denotes $\partial z^i/\partial x^\alpha$, and $f(x, z, p)$ $(p = \{p^i_\alpha\})$ is generally assumed continuous in all its arguments. The integrals $\int_a^b (1+(dz/dx)^2)^{1/2}dx$ and $\iint_G[(\partial z/\partial x^1)^2+(\partial z/\partial x^2)^2]dx^1dx^2$ are familiar examples of integrals of the form (1) in which $N=1$ in both cases, $\nu=1$ in the first case, and $\nu=2$ in the second case and the corresponding functions f are defined, respectively, by

$$f(x, z, p) = (1 + p^2)^{1/2}, \qquad f(x, z, p) = (p_1)^2 + (p_2)^2$$

where we have omitted the superscripts on z and p since $N=1$. The second integral is a special case of the *Dirichlet integral* which is defined in general by

$$D(z, G) = \int_G |\nabla z|^2 dx, \qquad f(x, z, p) = |p|^2 = \sum_{i,\alpha} (p^i_\alpha)^2. \tag{3}$$

Another example is the area integral

$$A(z, G) = \iint_G \left(\left[\frac{\partial(z^2, z^3)}{\partial(x^1, x^2)}\right]^2 + \left[\frac{\partial(z^3, z^1)}{\partial(x^1, x^2)}\right]^2 + \left[\frac{\partial(z^1, z^2)}{\partial(x^1, x^2)}\right]^2 \right)^{1/2} dx^1 dx^2 \tag{4}$$

which gives the area of the surface

$$z^i = z^i(x^1, x^2), \qquad (x^1, x^2) \in G, \quad i = 1, 2, 3. \tag{5}$$

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It is to be noticed that the area integral has the special property that it is invariant under diffeomorphisms (1-1 differentiable mappings, etc.) of the domain G into other domains. This is the first example of an *integral in parametric form.*

If f is of class C^n in the arguments (z, p), then I is a functional of class C^n defined on the Banach space $C^1(\overline{G})$. For each z and ζ, we define the *first variation* (if $n \geqq 1$) I_1 (z, ζ, G) by

$$(6) \qquad I_1(z, \zeta; G) = \phi'(0) \quad \text{where} \quad \phi(\lambda) = I(z + \lambda\zeta).$$

This is also the *Fréchet differential* of I. In case $z \in C^1(\overline{G})$ and furnishes a relative minimum or maximum among such functions with the same boundary values, then

$$(7) \qquad I_1(z, \zeta; G) = 0 \qquad \forall \zeta \in C^1(\overline{G}) \ni \zeta = 0 \quad \text{on } \partial G.$$

Let us first consider the case where $N = \nu = 1$. Then (7) becomes

$$(8) \qquad \int_a^b \{f_p[x, z(x), z'(x)] \cdot \zeta'(x) + f_z[x, z(x), z'(x)] \cdot \zeta(x)\} dx = 0$$
$$(G = (a, b)).$$

If we now assume that f and $z \in C^2(\overline{G})$, then we may integrate the first term of (8) by parts and obtain

$$(9) \qquad \int_a^b \zeta(x)[B(x) - A'(x)]dx = 0, \qquad A(x) = f_p[x, z(x), z'(x)],$$
$$B(x) = f_z[x, z(x), p(x)]$$

since $\zeta(a) = \zeta(b) = 0$. Thus we find that z satisfies *Euler's equation*

$$(10) \qquad df_p/dx = f_z \quad \text{or} \quad f_{pp}z'' + f_{pz}z' + f_{px} = f_z.$$

We notice that this equation is nonlinear (in general) and of the second order. It is, however, linear in z''; such an equation is called *quasi-linear.* The equation evidently becomes singular if $f_{pp} = 0$. Hence *regular* variational problems are those for which f_{pp} never vanishes; in that case, it is assumed that $f_{pp} > 0$ (so that f is convex in p) and this makes minimum problems more natural than maximum problems.

We notice that if z is of class C^2, then the equations (10) and (8) (assuming $\zeta = 0$ on ∂G) are equivalent. However, equation (8) makes sense even if z is merely of class C^1. In case z is known to be *at least* of class C^1 and to satisfy (8), we say that z is a *weak solution* of equation (10). It satisfies (10) in some distribution sense.

In the general case, if $z(=(z^1, \cdots, z^N))$ is of class $C^1(\overline{G})$ and furnishes a relative minimum to I among all vectors$\in C^1(\overline{G})$ which have the same boundary values on ∂G, then (7) holds. If we assume that $f\in C^2$, $z\in C^2(\overline{G})$ and $G\in C^1$, the technique of (8)–(10) shows that the vector z satisfies the system of Euler equations

$$\frac{\partial}{\partial x^\alpha} f_{p^i_\alpha} = f_{z^i} \quad \text{or} \quad f_{p^i_\alpha p^j_\beta} z^j_{,\alpha\beta} + f_{p^i_\alpha z^j} z^j_{,\alpha} + f_{p^i_\alpha x^\alpha} = f_{z^i}, \tag{11}$$

$$i = 1, \cdots, N;$$

a quasi-linear system. But if we merely assume $z\in C^1(\overline{G})$ and satisfies (7), we say that *z is a weak solution of the equations* (11).

Now if we define $\phi(\lambda)$ as in (6), we see that $\phi''(0)\geqq 0$, where $\phi''(0)$ is called *the second variation* and is given by

$$\begin{aligned} \phi''(0) &\equiv I_2(z, \zeta; G) \\ &= \int_G [f_{p^i_\alpha p^j_\beta} \zeta^i_{,\alpha} \zeta^j_{,\beta} + 2 f_{p^i_\alpha z^j} \zeta^i_{,\alpha} \zeta^j + f_{z^i z^j} \zeta^i \zeta^j] dx \ (\geqq 0) \end{aligned} \tag{12}$$

$$(\zeta = 0 \text{ on } \partial G).$$

From (12) it follows (see [**38**, pp. 10, 11]) that

$$f_{p^i_\alpha p^j_\beta} \lambda_\alpha \lambda_\beta \xi^i \xi^j \geqq 0 \qquad \forall\, \lambda = (\lambda_1, \cdots, \lambda_\nu), \quad \xi = (\xi^1, \cdots, \xi^N). \tag{13}$$

Regular problems are those for which the form (13) is positive for all $\lambda\neq 0$ and $\xi\neq 0$. *If $N=1$ or $\nu=1$*, the condition (13) implies the convexity of f as a function of p; if, also, the problem is regular, the forms

$$f_{p^i p^j} \xi^i \xi^j \ (\text{if } \nu = 1) \quad \text{and} \quad f_{p_\alpha p_\beta} \lambda_\alpha \lambda_\beta \ (\text{if } N = 1) \tag{14}$$

are required to be positive definite. The condition in (13) does not imply the convexity of f in all the p^i_α taken together in the general case. However, in all regular cases, the Euler equations form a *strongly elliptic* system in the sense defined by Nirenberg [**45**].

Now we observe that the equations (7) form a special case of the equations

$$\int_G \{\zeta^i_{,\alpha}(x) A^\alpha_i[x, z(x), \nabla z(x)] + \zeta^i(x) B_i[x, z(x), \nabla z(x)]\} dx = 0 \tag{15}$$

$$\forall\, \zeta \in C^1(\overline{G}) \ni \zeta = 0 \quad \text{on } \partial G,$$

where we have replaced $f_{p^i_\alpha}(x, z, p)$ by $A^\alpha_i(x, z, p)$ and f_{z^i} by B_i. If $z\in C^1(\overline{G})$ and satisfies (15), we say that z is a *weak solution* of the equations

$$\frac{\partial}{\partial x^\alpha} A_i^\alpha = B_i, \qquad i = 1, \cdots, N. \tag{16}$$

Of course we may consider equations of higher order and integrals involving derivatives of higher order. For that purpose we introduce the following notation: We let $\alpha = (\alpha_1, \cdots, \alpha_\nu)$, in which each α_λ is a nonnegative integer, be a *multi-index*, and denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_\nu, \qquad D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{(\partial x^1)^{\alpha_1} \cdots (\partial x^\nu)^{\alpha_\nu}}, \tag{17}$$

where $D^\alpha\phi = \phi$ if $\alpha = 0$. Then, if

$$I(z, G) = \int_G f[x, Dz(x)]dx, \qquad Dz(x) = \{D^\alpha z^i(x)\}, \tag{18}$$
$$0 \leqq |\alpha| \leqq m_i, \quad i = 1, \cdots, N,$$

the first variation of I is given by

$$I_1(z, \zeta; G) = \int_G \sum_{i=1}^N \sum_{|\alpha| \leqq m_i} D^\alpha \zeta^i(x) \cdot f_{p_\alpha^i}[x, Dz(x)]dx. \tag{19}$$

If each $z^i \in C^{m_i}(\overline{G})$ and $I_1(z, \zeta; G) = 0$ for all such $\zeta \ni D^\alpha \zeta^i = 0$ on ∂G for $0 \leqq |\alpha| \leqq m_i - 1$, we say that z is a *weak solution* of the equations

$$\sum_{|\alpha| \leqq m_i} (-1)^{|\alpha|} D^\alpha f_{p_\alpha^i} = 0. \tag{20}$$

We may replace $f_{p_\alpha^i}$ by A_i^α in (19) obtaining

$$\int_G \sum_{i=1}^N \sum_{|\alpha| \leqq m_i} D^\alpha \zeta^i \cdot A_i^\alpha[x, Dz(x)]dx = 0 \qquad \forall \zeta^i \in C_0^{m_i}(\overline{G}). \tag{21}$$

Finally, we may consider equations like (21) but in which m_i is replaced by some integer p_i and, in A_i^α, Dz stands for $\{D^\beta z^j\}$ where α and β satisfy proper inequalities. As an example of these more general equations, we give

$$\int_G \{\zeta_{,\alpha\beta}(x) a^{\alpha\beta}[x, z(x)] + \zeta_{,\alpha}(x) a^\alpha[x, z, \nabla z] \tag{22}$$
$$+ \zeta a[x, z, \nabla z, \nabla^2 z]\} dx = 0.$$

In general we shall not discuss equations more general than (21). In all these cases *the problem is to find conditions on the* A_i^α *in* (21) *which imply that any weak solution of* (21) *is a sufficiently differentiable solu-*

tion of the corresponding differential equations (or perhaps possesses some additional differentiability). Actually *we shall often allow our weak solutions to be in properly chosen Sobolev spaces*; we shall give more details later.

The general nature of the problem above is to conclude differentiability of solutions in addition to the minimum required to be solutions of the given equations. One of the earliest papers along this line was Hilbert's famous address [**18**] in which he proposed some 23 problems. Of these, the 19th was to show that the regular solutions of any analytic regular variational problem are analytic and the 23rd was, vaguely, to develop a theory of the calculus of variations. A problem more general than the 19th was solved by S. Bernstein in his famous memoir of 1904 [**3**] where he proved for the case of *any* analytic second order elliptic equation and solution of class C^3 on a domain in R^2 that the solution is analytic. His proof was rather long and many people presented simpler proofs, including H. Lewy [**26**], T. Rado [**48**], M. Gevrey [**14**], S. Bernstein himself [**4**], and others. E. E. Levi [**25**] proved the analyticity of the solutions of linear, analytic, elliptic equations of higher order in two dimensions. Bernstein's result was extended to solutions of class C_μ^2 on domains in R^n by E. Hopf [**21**] and was extended to higher order systems on R^n by I. Petrovsky [**47**]. C. B. Morrey, Jr. and L. Nirenberg [**40**] showed that regular solutions of analytic linear elliptic systems satisfying analytic Dirichlet boundary data along an analytic portion of the boundary of a domain could be extended analytically across that portion. This result was extended to analytic nonlinear systems in 1957/58 by A. Friedman [**13**] and myself [**35**], independently, using different methods. Finally, in my book [**38**], I treat the case of general regular boundary conditions (in the sense of Agmon, Douglis and Nirenberg [**1**]).

A somewhat different series of generalizations of Bernstein's result was begun by L. Lichtenstein [**27**] when he showed in 1912 that a solution of class C^2 on a domain in R^2 of an analytic *variational problem* (of the type in (1) with $\nu=2$, $N=1$) is analytic. This result was extended in 1929 by E. Hopf [**20**] to the case where the solution was required only to be of class C_μ^1 (of class C^1 with derivatives satisfying Hölder conditions of exponent μ) for some μ with $0<\mu<1$. Morrey [**30**] extended this result further in January 1938 to the case where the solution was required merely to satisfy a Lipschitz condition.

All of the results mentioned so far assumed the existence of a solution having certain differentiability properties. Hilbert [**19**] was probably the first to use variational methods to prove the existence of a harmonic function having given boundary values on a domain of a fairly general type.

Tonelli, Lebesgue, Fréchet, and many others developed the so-called "direct methods" of the calculus of variations. The idea of these methods is to show

(i) that the integral to be minimized is lower-semicontinuous with respect to some kind of convergence,

(ii) that it is bounded below in some class of "admissible functions," and

(iii) that there is a "minimizing sequence," i.e. a sequence for which the integral tends to its infimum, which converges in the sense required to some admissible function.

Tonelli exploited and popularized these methods in a series of papers and a book [56]–[62]. He applied the methods to many one dimensional (i.e. $\nu = 1$) problems and to some two dimensional ones. For the one dimensional problems, he found it expedient to allow absolutely continuous functions as admissible and to use uniform convergence. This comes about roughly as follows: Suppose that

$$f(x, z, p) \geqq m|p|^r - K, \qquad r > 1,\ m > 0, \tag{23}$$

(which is not unreasonable since f is convex in p). Then

$$\int_a^b |z_n'(x)|^r dx \leqq L \tag{24}$$

in any minimizing sequence $\{z_n\}$. From (24), one sees from the Hölder inequality that any minimizing sequence is equicontinuous. It is also true for any uniformly convergent subsequence of a minimizing sequence that the limit function is also absolutely continuous with

$$\int_a^b |z'(x)|^r dx \leqq \liminf_{n\to\infty} \int_a^b |z_n'(x)|^r dx.$$

(This type of convergence turns out to be weak convergence in the Sobolev-space $H_r^1(a, b)$ for the case where $\nu = 1$.)

For the two dimensional problems, he defined functions which he called absolutely continuous but then found it expedient to require f to satisfy (23) with $r > 2$ (where $|p| = (p_1^2 + p_2^2)^{1/2}$) in order to obtain equicontinuous minimizing sequences. Actually if he assumed that $f(x, z, p) \geqq f(x, z, 0) = 0$ for all (x, z, p), he could allow $r = 2$. However, this leaves a gap, $1 < r < 2$, in which Tonelli was unable to obtain a general theorem. Moreover, if one considers integrals in which $\nu > 2$, one soon finds that one must assume $r>\nu$ in order to assure that minimizing sequences are equicontinuous. To see this, it is only necessary to observe that the functions

$$\log\log(1+|x|^{-1}), \qquad |x|^{-h}, \qquad 0<|x|<1$$

are limits of C^1 functions z_n in which

$$\int_G |\nabla z_n|^\nu dx \quad \text{and} \quad \int_G |\nabla z_n|^k dx \qquad \text{for } k<\nu/(h+1)$$

are uniformly bounded over the unit ball G.

In order to get a more complete existence theory, the writer and Calkin [4], [31], [32], [33] found it expedient to allow as admissible, functions which are still more general than Tonelli's functions and to allow correspondingly more general types of convergence and boundary values. These new spaces of functions can now be identified with the so-called Sobolev spaces $H_p^1(G)$ (or $W_p^1(G)$) which are now used by many writers in many connections. Unfortunately the minimizing function shown to exist was known only to be in one of these general spaces and hence was not known to be continuous, let alone of class C^2! However, A. Haar [17] (see also T. Rado [49]) had proved previously that if $\nu=2$, $N=1$, and $f=f(p)$, there exists a unique minimizing function z which is defined on a strictly convex domain G and which satisfies a Lipschitz condition there, provided the given boundary values satisfy a certain "three point condition." Combining this with my January 1938 result mentioned above, we obtain an existence and differentiability theorem for the Euler equation of any analytic variational problem of this type. During the year 1937/38, I was able to show for a wide class of integrals in the case $\nu=2$, N arbitrary, that any minimizing vector satisfies a "Dirichlet growth" condition and that any solution satisfying this condition is of class C_μ^n ($n\geqq 3$, $0<\mu<1$) if f is and is analytic or C^∞ if f is. I lectured on this work in the seminar of Professor Marston Morse during the spring of 1938. Notes on these lectures were prepared by H. Busemann and are still to be found in the library of the Institute for Advanced Study under his name.

We must now define the so-called Sobolev spaces $H_p^m(G)$. Actually, these functions were used by G. C. Evans [11] in his work on potential theory beginning in 1920. Others had used these functions well before Sobolev [55] proved his famous result in 1938. Many of the standard results concerning such functions were proved in the papers by Calkin and Morrey cited above. Since distributions and partitions of unity had not been introduced, different terminology was used of course.

DEFINITION. A function z is *of class* $H_r^m(G)$, $r \geqq 1$, G open, $\Leftrightarrow z$ is of class $L_r(G)$ and there exist functions p_α, $0 \leqq |\alpha| \leqq m$, each also of class $L_r(G)$ such that

$$\int_G g(x) p_\alpha(x) dx = (-1)^{|\alpha|} \int_G z(x) D^\alpha g(x) dx \tag{25}$$

for every $g \in C^\infty(G)$ (C_c^∞, compact support on G).

It is clear that the functions p_α are uniquely determined up to null functions and that if z is of class $H_r^m(G)$ and $z^* = z$ almost everywhere on G, then z^* is of class $H_r^m(G)$ and the same functions p_α will do for z^*.

DEFINITION. As in the case of the L_r spaces, the *elements* of the *space* $H_r^m(G)$ are the classes of equivalent functions *of class* $H_r^m(G)$. We denote the classes of equivalent functions p_α by $z_{,\alpha}$ or $D^\alpha z$ and call them the *distribution derivatives* of the element z.

It turns out that $z \in H_r^m(G) \Leftrightarrow z$ and its distribution derivatives of order $\leqq m-1 \in H_r^1(G)$, etc. Of course we may regard an element $z \in H_r^m(G)$ as a distribution and then the distribution corresponding to $z_{,\alpha}$ would be the corresponding derivative of z in the distribution sense. The spaces $H_r^m(G)$ have been defined for all *real* m. We may allow real or complex functions z.

The following theorems about these spaces are known (see, for instance, [**38**, Chapter 3]).

The spaces $H_r^m(G)$ *are Banach spaces if we define*

$$\|z\|_r^m = \sum_{|\alpha| \leqq m} \|z_{,\alpha}\|_r^0, \tag{26}$$

for instance. If $r = 2$, *they are Hilbert spaces if we define*

$$(z, w)^m = \sum_{|\alpha| \leqq m} (z_{,\alpha}, w_{,\alpha})^0. \tag{27}$$

If $r > 1$, *the spaces are reflexive.*

If $z \in C^{m-1}(G)$ *and all the* $D^\alpha z$ *with* $|\alpha| \leqq m-1$ *are locally Lipschitz on* G, *then* z *is of class* $H_r^m(G) \Leftrightarrow$ *its norm (as defined in* (26)) *is bounded.*

Functions of class $C^\infty(G)$ *with finite norm in* (26) *are dense in* $H_r^m(G)$; *if* G *is "strongly Lipschitz" (regular in the sense of Calderon) (see* [**38**, *p.* 72]), *the restrictions to* G *of functions of class* $C_c^\infty(D)$, *where* $D \supset \bar{G}$, *are dense in* $H_r^m(G)$.

Suppose $x = x(y)$ *is a regular mapping of class* C^m ($m \geqq 1$) *of* H *onto* G *in which all the derivatives of* $x(y)$ *and of the inverse mapping* $y(x)$ *are bounded. Then if* u *is of class* $H_r^m(G)$ *and*

$$v(y) = u[x(y)], \tag{28}$$

$v \in H_r^m(H)$ and the derivatives of v are obtained as usual a.e.

Each element $u^* \in H_r^1(G)$ contains a representative u which is absolutely continuous along almost all lines parallel to each axis and which is such that if v is defined by (28), *then v has the same property and is, of course of class $H_r^1(H)$. If u^* contains a continuous representative, then that is unique and can be taken as u.*

(Sobolev [55].) *If G is Lipschitz (see* [38, *p.* 77]) *and $z \in H_r^1(G)$ with $1 \leqq r < \nu$, then $z \in L_s(G)$ and there exists a constant $C(\nu, r, G)$ such that*

$$\|z\|_s^0 \leqq C \cdot \|z\|_r^1, \qquad s = \nu r/(\nu - r).$$

DEFINITION. We define the space $H_{r0}^m(G)$ to be the closure in $H_r^m(G)$ of the set $C_c^\infty(G)$.

We have the following theorems.

(Poincaré's inequality.) *Suppose $G \subset B(x_0, R)$ and $z \in H_{r0}^m(G)$. Then*

$$\int_G |\nabla^k z(x)|^r dx \leqq r^{k-m} R^{(m-k)r} \int_G |\nabla^m z(x)|^r dx, \qquad 0 \leqq k \leqq m.$$

(a) *Suppose $u \in H_{r0}^m(G)$, $V(x) = u(x)$ for $x \in G$, and $V(x) = 0$ elsewhere. Then $V \in H_r^m(R_\nu)$ and $V \in H_{r0}^m(\Delta)$ for any open $\Delta \supset G$. Moreover $D^\alpha V(x) = D^\alpha u(x)$ on G and $D^\alpha V(x) = 0$ for $x \in R_\nu - G$ if $0 \leqq |\alpha| \leqq m$ (a.e.).*

(b) *Suppose $u \in H_r^m(G)$, $D \subset G$, $v \in H_r^m(D)$, $v - u|_D \in H_{r0}^m(D)$, $U(x) = v(x)$ on D, and $U(x) = u(x)$ on $G - D$. Then $U \in H_r^m(G)$, $U - u \in H_{r0}^m(G)$, and $D^\alpha U(x) = D^\alpha v(x)$ on D and $D^\alpha U(x) = D^\alpha u(x)$ on $G - D$ (a.e.).*

$z_n \rightharpoonup$ (tends weakly) z in $H_r^m(G) \Leftrightarrow D^\alpha z_n \rightharpoonup D^\alpha z$ in $L_r(G)$ for each α with $|\alpha| \leqq m$.

The first of these results shows that

$$\|z\|_{r0}^m = \left[\int_G |\nabla^m z|^r dx \right]^{1/r}$$

is equivalent to the norm $\|z\|_r^m$ for $z \in H_{r0}^m(G)$.

The change of variable theorem enables us to define the spaces H_r^m on manifolds of class C^m (or even C_1^{m-1}); of course there are no particularly natural norms on these spaces although many satisfactory norms can be defined which are topologically equivalent if the manifold is compact. We have the following results.

Suppose G is strongly Lipschitz and $m \geqq 1$. Then bounded subsets of $H_r^m(G)$ are conditionally sequentially compact as subsets of $H_r^{m-1}(G)$. If $u_n \rightharpoonup u$ in $H_r^m(G)$ then $u_n \rightarrow u$ in $H_r^{m-1}(G)$. If $m = 1$, G may be merely

Lipschitz (see [38, p. 77]). The theorem above is true for any bounded domain if we replace the spaces $H_r^m(G)$ and $H_r^{m-1}(G)$, respectively by $H_{r0}^m(G)$ and $H_{r0}^{m-1}(G)$.

If G is (bounded and) of class C_1^{m-1} the functions $u \in C_1^{m-1}(\bar{G})$ are dense in any space $H_r^m(G)$ with $r \geqq 1$ and there is a bounded operator B from $H_r^m(G)$ into $H_r^{m-1}(\partial G)$ such that $Bu = u|_{\partial G}$ whenever $u \in C_1^{m-1}(\bar{G})$. If $u_n \rightharpoonup u$ in $H_r^m(G)$, $Bu_n \rightharpoonup Bu$ in $H_r^{m-1}(\partial G)$. If $r > 1$, the mapping B is compact.

We now present a sample lower-semicontinuity theorem. We need the fact that a convex function $f(\xi)$ satisfies

$$f(\xi) \geqq f(\xi_0) + f_{,\alpha}(\xi_0)(\xi^\alpha - \xi_0^\alpha) \qquad \forall \xi.$$

Suppose $f = f(x, z, p)$ and the $f_{p_\alpha^i}$ are continuous with $f(x, z, p) \geqq 0$ for all (x, z, p), suppose f is convex in p for each (x, z), and suppose $z_n \rightharpoonup z$ in $H_1^1(D)$ for each $D \subset\subset G$. Then

$$I(z, G) \leqq \liminf I(z_n, G).$$

Proof. Choose $D \subset\subset G$. The weak convergence in $H_1^1(D')$ for each $D' \subset\subset G$ implies the strong convergence of z_n to z in $L_1(D)$. By choosing a subsequence, still called z_n, in which $I(z_n, G) \to$ its former lim inf, we may assume that $z_n(x) \to z(x)$ a.e. on D. We now suppose $I(z, D) < +\infty$. Then, for each $\epsilon > 0$, there is a compact set $S \subset D$ on which representatives of z and ∇z are continuous, on which z_n converges uniformly to z, and which is such that

$$I(z, S) > I(z, D) - \epsilon$$

(if $I(z, D) = +\infty$, we may take $I(z, S) > M$, arbitrary). Then from the convexity of f, we conclude that, for $x \in S$,

$$\begin{aligned} f[x, z_n(x), \nabla z_n(x)] \geqq\ & f[x, z_n(x), \nabla z(x)] + f_p[x, z(x), \nabla z(x)] \\ & \cdot [\nabla z_n(x) - \nabla z(x)] \\ & + \{f_p[x, z_n(x), \nabla z(x)] - f_p[x, z(x), \nabla z(x)]\} \\ & \cdot [\nabla z_n(x) - \nabla z(x)]. \end{aligned}$$

The weak convergence implies the weak convergence of ∇z_n to ∇z in $L_1(S)$ which implies, in turn, that

$$\int_S f_p[x, z(x), \nabla z(x)] \cdot [\nabla z_n(x) - \nabla z(x)] dx \to 0.$$

The uniform convergence of z_n to z on S and the uniform boundedness of the $L_1(S)$ norms of ∇z_n and ∇z implies that

$$\int_S \{f_p[x, z_n, \nabla z] - f_p[x, z, \nabla z]\} \cdot [\nabla z_n - \nabla z]dx \to 0.$$

Hence

$$I(z, D) - \epsilon \text{ (or } M) \leqq \lim_{n\to\infty} \int_S f[x, z_n, \nabla z]dx \leqq \liminf I(z_n, G).$$

We now present a sample existence theorem.

Suppose that

(i) f *and* $f_{p_\alpha^i}$ *are continuous in their arguments*;

(ii) f *is convex in* p *for each* (x, z);

(iii) *for all* (x, z, p)

$$f(x, z, p) \geqq m\,|\,p\,|^k, \qquad m > 0, \quad k > 1;$$

(iv) F^* *is a nonempty family of vector functions which is compact with respect to weak convergence in* $H_k^1(G)$;

(v) F *is a family, closed under weak convergence in* $H_k^1(G)$, *such that each* z *in* F *coincides with some* z^* *in* F^* *on* ∂G (*i.e.* $z - z^* \in H_{k0}^1(G)$);

(vi) $I(z_0, G) < +\infty$ *for some* $z_0 \in F$; *and*

(vii) G *is bounded.*

Then $I(z, G)$ *takes on its minimum for some* z *in* F.

Since we have not made any assumptions about G other than that it is bounded and since the admissible functions are not necessarily continuous, the most convenient way to specify the boundary values of a function z is to state that $z - z^* \in H_{k0}^1(G)$ for some given z^*. Thus the family F^* defines the boundary values being allowed, so to speak. Of course F^* could consist of a single function z^*. In case G is at least Lipschitz, then the family F^* could be replaced by a family of functions defined on ∂G and then each z in F would be required to be such that $Bz \in F^*$. This change would make little difference in the proof below if F^* were closed in $L_k(\partial G)$.

Proof. Let $\{z_n\}$ be a minimizing sequence; we may assume that $I(z_n, G) \leqq I(z_0, G) = M$ for every n. Using (iii), we find that

$$\int_G |\,\nabla z_n\,|^k dx \leqq L = M/m, \qquad n = 1, 2, \cdots.$$

From (v), we know that there exists a $z_n^* \in F^* \ni w_n = z_n - z_n^* \in H_{k0}^1(G)$ for each n. From (iv), we may extract a subsequence, still called $\{n\}$ such that $z_n^* \rightharpoonup$ some $z^* \in F^*$. We also see that $\int_G |\,\nabla w_n\,|^k dx$ and hence (since $w_n \in H_{k0}^1(G)$) $\|w_n\|_k^1$ is uniformly bounded. Hence, from the reflexivity we conclude that, for a further subsequence, still called

$\{n\}$, $w_n \rightharpoonup w$ in $H^1_k(G)$. Since $H^1_{k0}(G)$ is a closed linear manifold in $H^1_k(G)$ we see that $w \in H^1_{k0}(G)$. Thus $z_n \rightharpoonup z = z^* + w$ and z is the desired solution.

If f satisfies no additional conditions, it is not necessarily true that a minimizing function has any further continuity and differentiability properties. Rather than stating the most general conditions ensuring further differentiability, we state two sets of conditions on the integrand function f under which differentiability results have been obtained. We require f first to satisfy the

COMMON CONDITION. *$f \in C^2_\mu$ in its arguments or f and $f_p \in C^{n-1}_\mu$ for some $n \geqq 3$ and some μ with $0 < \mu < 1$.*

Besides this condition, we require that f satisfy one of the following sets for all (x, z, p):

$$\text{(A)}\quad \begin{aligned} & mV^k - K \leqq f(x, z, p) \leqq MV^k \\ & |f_p|^2 + |f_{px}|^2 + |f_z|^2 + |f_{zx}|^2 \leqq M_1^2 V^{2k-2} \\ & |f_{pz}|^2 + |f_{pp}|^2 \leqq M_1^2 V^{2k-4} \\ & m_1 V^{k-2} |\pi|^2 \leqq \sum f_{p^i_\alpha p^j_\beta} \pi^i_\alpha \pi^j_\beta \end{aligned}$$

$$k > 1, \quad 0 < m \leqq M, \quad 0 < m_1 \leqq M_1, \quad V = (1 + |z|^2 + |p|^2)^{1/2}$$

$$(|f_{pp}|^2 = \sum (f_{p^i_\alpha p^j_\beta})^2, \qquad |f_{pz}|^2 = \sum (f_{p^i_\alpha z^j})^2, \qquad |\pi|^2 = \sum (\pi^i_\alpha)^2, \text{etc.}$$

(A′) Same as (A) except that $f = f(x, p)$, $V = (1 + |p|^2)^{1/2}$.

$$\text{(B)}\quad \begin{aligned} & mV^k - K \leqq f(x, z, p) \leqq MV^k \\ & |f_z|^2 + |f_{zx}|^2 + |f_{zz}|^2 \leqq M^2(R) V^{2k} \\ & |f_p|^2 + |f_{pz}|^2 + |f_{px}|^2 \leqq M^2(R) V^{2k-2} \\ & m_1(R) V^{k-2} |\pi|^2 \leqq \sum f_{p^i_\alpha p^j_\beta} \pi^i_\alpha \pi^j_\beta \leqq M_1(R) V^{k-2} |\pi|^2 \\ & k \geqq \nu, \quad 0 < m \leqq M, \quad V = (1 + |p|^2)^{1/2} \\ & 0 < m_1(R) \leqq M_1(R) \quad \text{for } |x|^2 + |z|^2 \leqq R^2. \end{aligned}$$

We notice that (B) reduces to (A′) if f does not depend on z, except for the R-condition which is somewhat meaningless since we always assume G bounded. To see the difference between (A) and (B), we notice that the f defined by

$$f(x, z, p) = [1 + a^{\alpha\beta}_{ij}(x, z) p^i_\alpha p^j_\beta]^{k/2}$$

satisfies the conditions (B) but not (A) if $k \geqq \nu$ and the a's$\in C^2_\mu$ or C^{n-1}_μ if $n \geqq 3$ and the quadratic form is positive definite. We think of k as the *degree* of f in (p, z) or p.

The first step in the proof of differentiability of the solutions is to note the following continuity properties of the solutions:

If $k \geqq \nu$, the minimizing functions are Hölder continuous on interior domains.

PROOF. If $k > \nu$, we see by the Hölder inequality that

$$(29)\quad \int_{B(x_0,r)} |\nabla z|\, dx \leqq C(\nu, k) \|\nabla z\|^0_k r^{\nu-1+\mu}, \quad \mu = 1 - \nu/k, \quad B(x_0, r) \subset G.$$

If $\nu = k$ and z is minimizing, we see from either (A) or (B) that

$$(30)\quad m|p|^k - K' \leqq f(x, z, p) \leqq M|p|^k + L', \qquad 0 < m \leqq M$$

so that if z is minimizing

$$(31)\quad \begin{aligned} m\int_{B(x_0,r)} |\nabla z|^\nu dx - K'\gamma_\nu r^\nu &\leqq I[z, B(x_0, r)] \leqq I[Z, B(x_0, r)] \\ &\leqq L'_{\gamma_\nu} r^\nu + M \int_{B(x_0,r)} |\nabla Z|^k dx \end{aligned}$$

$$(\gamma\nu = \text{meas } B(0, 1))$$

where Z is any vector function such that $Z - z \in H^1_k[B(x_0, r)]$. Setting

$$(32)\quad Z(s, \theta) = \bar{z} + (s/r)[z(r, \theta) - \bar{z}], \qquad \theta \in \partial B(0, 1),$$

$$(\bar{z} = \text{av } z \text{ on } \partial B(x_0, r))$$

it is easy to see, using the fact that

$$|\nabla Z(s, \theta)|^2 = Z_s^2 + s^{-2}|\nabla_\theta Z|^2 = r^{-2}\{|z(r, \theta) - \bar{z}|^2 + |\nabla_\theta z(r, \theta)|^2\},$$

that (setting $d\sum$ as $(\nu - 1)$-area element on $\partial B(0, 1)$)

$$(33)\quad \begin{aligned} \phi(r) \equiv \int_{B(x_0,r)} |\nabla z|^\nu dx &\leqq m^{-1}\gamma_\nu(K' + L')r^\nu \\ &+ C\int_{\partial B(x_0,r)} |\nabla_\theta z(r, \theta)|^\nu d\sum \leqq C_1 r\phi'(r) + C_2 r^\nu. \end{aligned}$$

It follows easily from (33) that

$$(34)\quad \int_{B(x_0,r)} |\nabla z|^\nu dx \leqq (\phi(R) + C_3 R^\nu)(r/R)^{\nu\mu},$$

$$0 \leqq r \leqq R, \; B(x_0, R) \subset G,$$

from which (29) follows easily where, however, C depends on the distance $d(x_0, \partial G)$ of x_0 from ∂G and μ is that in (34). The result follows from the following theorem about Sobolev spaces.

If z satisfies a condition like (29) *where C may depend on $d(x_0, \partial G)$ and μ is any number with $0<\mu<1$, then $z \in C_\mu^0(\bar{D})$ on any compact set $\bar{D} \subset G$ (see* [**38**, *p.* 79]).

The next step in the differentiability program is stated in the following theorem.

Suppose f satisfies (A) *or* (A′) *for some $k>2$ or satisfies* (B) *with $k>\nu$. Then $I(z) \equiv I(z, G)$ is of class C^2 over the space $H_k^1(G)$* (G bounded). *If f satisfies* (A) *or* (A′) *with $1<k\leqq 2$, then $I(z)$ is of class C^1 over $H_k^1(G)$. In all cases, if z is minimizing, the first differential (variation) of I vanishes at z.*

In other words, if z is minimizing z satisfies (7). We call any function z for which the first variation vanishes an *extremal* whether z is minimizing or not and we also call such a z a *weak solution* of the corresponding Euler equations.

So far, we have shown the existence only of a minimizing function for I but this was done for rather general functions f. Recently Palais and Smale [**46**] have found a modification of the Morse theory which is applicable to a wide variety of variational problems. In their theory an extremal is just a *critical point* for the integral.

To illustrate the next step in the differentiability program, we sketch the proof of our January 1938 result mentioned above in which we assume that the extremal z satisfies a Lipschitz condition and that $f=f(p, q)$, that $\nu=2$, that $f \in C_\mu^2$, and f is regular (cf. the middle inequality in A or B), i.e. that

$$f_{pp}(p, q)\lambda^2 + 2f_{pq}\lambda\mu + f_{qq}\mu^2 > 0 \quad \text{if } \lambda^2 + \mu^2 > 0. \tag{35}$$

Then z satisfies

$$\begin{gathered} \iint_G [\zeta_x(x, y)A(x, y) + \zeta_y(x, y)B(x, y)]dxdy = 0 \qquad \forall \zeta \in \mathrm{Lip}_c(G) \\ A(x, y) = f_p[z_x(x, y), z_y(x, y)], \qquad B(x, y) = f_q[\text{same}]. \end{gathered} \tag{36}$$

Choose $\zeta \in \mathrm{Lip}_c(G)$ and define (for h small, $\neq 0$)

$$\begin{aligned} \zeta_h(x, y) &= h^{-1}[\zeta(x - h, y) - \zeta(x, y)], \\ z_h(x, y) &= h^{-1}[z(x + h, y) - z(x, y)]. \end{aligned} \tag{37}$$

If we replace ζ in (10) by ζ_h, make the obvious change of variable in the terms involving $\zeta_x(x-h, y)$, etc., we obtain the equation

$$(38)\quad h^{-1}\int\int_G (\zeta_x \Delta A + \zeta_y \Delta B) dxdy = 0,\ \ \Delta A = A(x+h, y) - A(x, y).$$

Since z is Lipschitz, its partial derivatives exist almost everywhere and are bounded. Thus, for almost all (x, y), we may express ΔA and ΔB in terms of z_{hx} and z_{hy} using the integral form of the theorem of the mean. When this is done, (38) becomes

$$(39)\qquad \int\int_G [\zeta_x(a_h z_{hx} + b_h z_{hy}) + \zeta_y(b_h z_{hx} + c_h z_{hy})] dxdy = 0$$

where a_h, b_h, c_h are bounded and measurable independently of h and

$$(40)\qquad a_h(x, y) = \int_0^1 f_{pp}[z_x(x, y) + t\Delta z_x, z_y(x, y) + t\Delta z_y] dt$$

and b_h and c_h are obtained similarly from f_{pq} and f_{qq}. Also, from (35) and the boundedness of z_x and z_y, we conclude that there are numbers m and M, independent of h, such that

$$(41)\quad m(\lambda^2 + \mu^2) \leqq a_h\lambda^2 + 2b_h\lambda\mu + c_h\mu^2 \leqq M(\lambda^2 + \mu^2),\quad 0 < m \leqq M.$$

Now, we choose $\delta > 0$ and define

$$\begin{aligned} \phi(x, y) &= 1, & (x, y) &\in G_{2\delta},\\ &= 1 - \delta^{-1} d(x, y, G_{2\delta}), & (x, y) &\in G_\delta - G_{2\delta},\\ &= 0, & (x, y) &\in G - G_\delta, \end{aligned}$$

$$G_a = \{(x, y) \mid B(x, y; a) \subset G\}$$

$$\zeta = \phi Z_h, \qquad Z_h = \phi z_h, \qquad 0 < |h| < \delta.$$

Then, for each h, ζ and Z_h are Lipschitz and have support on $\overline{G}_\delta$. We also have (a.e.)

$$\zeta_x = \phi(Z_{hx} + \phi_x z_h), \qquad \zeta_y = \phi(Z_{hy} + \phi_y z_h),$$

$$\phi z_{hx} = Z_{hx} - \phi_x z_h, \qquad \phi z_{hy} = Z_{hy} - \phi_y z_h.$$

Substituting these results into (39), we find that

$$\int\int_{G_\delta} [a_h Z_{hx}^2 + 2b_h Z_{hx} Z_{hy} + c_h Z_{hy}^2) - z_h^2(a_h\phi_x^2 + 2b_h\phi_x\phi_y + c_h\phi_y^2)] dxdy = 0.$$

From (41) and the definitions of ϕ and Z_h, we see that

$$\iint_{G_{2\delta}} (z_{hx}^2 + z_{hy}^2)dxdy \leqq \frac{M}{m}\delta^{-2}\iint_{G_\delta} z_h^2 dxdy. \tag{42}$$

Since z is Lipschitz, the right side of (42) is bounded independently of h for each $\delta > 0$ so the left side is also. Thus, for a sequence of $h \to 0$, $z_{hx} \rightharpoonup u$ and $z_{hy} \rightharpoonup v$ in $L_2(G_{2\delta})$ for some u and v. But now $z_h(x, y) \to z_x(x, y) = p(x, y)$ a.e. and boundedly. Now if $\psi \in C_c^\infty(G_{2\delta})$

$$\begin{aligned} \iint_{G_{2\delta}} \psi z_{hx} dxdy &= -\iint_{G_{2\delta}} \psi_x z_h dxdy, \\ \iint_{G} \psi z_{hy} dxdy &= -\iint_{G} \psi_y z_h dxdy. \end{aligned} \tag{43}$$

We may let $h \to 0$ in (43) (thru the special subsequence) and get

$$\begin{aligned} \iint_{G_{2\delta}} \psi u dxdy &= -\iint_{G_{2\delta}} \psi_x p dxdy, \\ \iint_{G_{2\delta}} \psi v dxdy &= -\iint_{G_{2\delta}} \psi_y p dxdy. \end{aligned} \tag{44}$$

Thus $p \in H_2^1(G_{2\delta})$ and $u = p_x$, $v = p_y$. Since a_h, b_h, c_h tend a.e. and boundedly to their respective limits

$$a = f_{pp}[z_x(x, y), z_y(x, y)], \qquad b = f_{pq}[\ ], \qquad c = f_{qq}[\ ]; \tag{45}$$

$\zeta_x a_h \to \zeta_x a$, etc., strongly in $L_2(G_{2\delta})$ if $\zeta \in \mathrm{Lip}_c(G_{2\delta})$. Thus we may let $h \to 0$ in (39) and conclude that p satisfies the limiting equations

$$\iint_{G_{2\delta}} [\zeta_x(au_x + bu_y) + \zeta_y(bu_x + cu_y)]dxdy = 0. \tag{46}$$

Likewise $q = z_y$ is seen to $\in H_2^1(G_{2\delta})$ and to satisfy (46).

Now, I proved before the war that solutions of such equations are Hölder continuous on interior regions if $\nu = 2$. Using that information, we have p and q satisfying the limit equations (46) in which a, b, and c are Hölder continuous. From an old theorem of Lichtenstein, it follows that p and $q \in C_\mu^1$ so that $z \in C_\mu^2$. Higher differentiability follows by repetition.

As was mentioned above the author carried the program above through for the cases where f satisfies the conditions (B) with $k = \nu = 2$ and N arbitrary during the year 1937/38. The author reported on this in an invited address to the Society at its meeting in Pasadena on December 2, 1939 [32]. The long manuscript for this work was approved for publication by the University of California Press in

1939 and appeared in December 1943 [33]. The writer used these results in his solution of the problem of Plateau on a Riemannian manifold [34].

Attempts to extend these results to (nonlinear) cases where $\nu>2$ met with no success until De Giorgi [9] and Nash [42] independently showed that a solution u in $H_2^1(G)$ of an equation (like (46)) of the form

$$\begin{gathered}\int_D \zeta_{,\alpha}a^{\alpha\beta}u_{,\beta}dx = 0,\\ \forall\zeta \in \overset{1}{H}_{20}(D), \qquad m|\lambda|^2 \leqq a^{\alpha\beta}(x)\lambda_\alpha\lambda_\beta \leqq M|\lambda|^2,\end{gathered} \tag{47}$$

with the $a^{\alpha\beta}$ bounded and measurable, is Hölder continuous on interior domains. A simplification of this work due to Moser [41] and the development of other techniques enabled the author and a student E. R. Buley [36], [37] and simultaneously O. A. Ladyzenskaya and N. Uraltseva [22], [23], [24] to prove that the solutions (of a variational problem) $\in C_\mu^n$ in case the integrand function f satisfies the conditions (A) or (A′) with $N=1$ and $k>1$ or (B) with $k\geqq\nu$ and $N=1$, but ν arbitrary in both cases. Ladyzenskaya and Uraltseva obtained also the differentiability results in the cases (B) where $2\leqq k<\nu$, provided the solution was known to be bounded. They also noticed that the results could be carried over to (weak) solutions z in $H_k^1(G)$ of equations of the form

$$\int_G [\zeta_{,\alpha}A^\alpha(x, z, \nabla z) + \zeta B(x, z, \nabla z)]dx = 0 \qquad \forall\zeta \in \mathrm{Lip}_c G$$

provided the $A^\alpha(x, z, p)$ and $B(x, z, p)$ satisfy conditions analogous to (A), (A′), and (B), namely,

$$\begin{aligned}&|A|^2 + |A_x|^2 + |B|^2 + |B_x|^2 \leqq M_1^2V^{2k-2}\\ &|A_z|^2 + |A_p|^2 + |B_z|^2 + |B_p|^2 \leqq M_1^2V^{2k-4}\\ &m_1V^{k-2}|\pi|^2 \leqq \overset{\alpha}{A}{}_{ip_\beta^j}\overset{i}{\pi}_\alpha\overset{j}{\pi}_\beta\\ &k > 1, \qquad m_1 > 0, \qquad V = (1 + |z|^2 + |p|^2)^{1/2}\end{aligned} \tag{C}$$

(C′) The same as (C) with $A=A(x, p)$, $B=0$, $V(1+|p|^2)^{1/2}$, $k>1$.

$$\begin{aligned}&|B|^2 + |B_z|^2 + |B_x|^2 \leqq M_1^2(R)V^{2k}\\ &|A|^2 + |A_z|^2 + |A_x|^2 + |B_p|^2 \leqq M_1^2(R)V^{2k-2}\\ &m_1(R)V^{k-2}|\pi|^2 \leqq \overset{\alpha}{A}{}_{ip_\beta^j}\overset{i}{\pi}_\alpha\overset{j}{\pi}_\beta, \quad |A_p| \leqq M_1(R)V^{k-2}\\ &k \geqq \nu, \quad m_1(R) > 0, \quad V = (1 + |p|^2)^{1/2}, \quad |x|^2 + |z|^2 \leqq R^2\end{aligned} \tag{D}$$

In all cases, it is assumed that the A_i^α and $B_i \in C_\mu^1$ if $n=2$ or the $A_i^\alpha \in C_\mu^{n-1}$ and $B_i \in C_\mu^{n-2}$ if $n \geqq 3$. And, of course, it is not assumed that $A_{ip_\beta^j}^{\alpha j} = A_{jp_\alpha^i}$. The condition on the quadratic form in π really is equivalent to a condition on its symmetric part

$$\tfrac{1}{2}(A^\alpha_{ip^j_\beta} + A^\beta_{jp^i_\alpha})\pi^i_\alpha \pi^j_\beta \geqq m_1(R) V^{k-2} |\pi|^2.$$

It is to be noticed that the results above for the cases $\nu > 2$ all require $N=1$.

So far, I have spoken only of variational problems in nonparametric form. It will be recalled that an integral in parametric form is one in which the integral is invariant under diffeomorphisms (possibly with positive jacobian) in the independent variables. Thus one looks for solutions which are "geometric surfaces" in some sense, each of which might have many parametric representations which might be vector functions giving the same value to the integral. It turns out that the function f for such a problem must be independent of x, must have $N \geqq \nu$, and must depend only on the $\nu \times \nu$ determinants in the $\{p_\alpha^i\}$ matrix. The area integral in (4) has this property. Such functions cannot be convex in the p_α^i although they do satisfy the condition (13) if the rank of the p matrix is ν. Also, in general, f is singular along the manifold in the p-space where the p matrix is of rank $< \nu$.

The problem of Plateau (the problem of finding a smooth minimal surface of least area) for surfaces which are images of the unit disc and bounded by a single given contour was solved in 1930/31 simultaneously by J. Douglas [**10**] and T. Rado [**50**]. These men and others, including McShane [**29**] and Courant [**7**] solved the problem in the 1930's for surfaces of more general topological types and/or bounded by more than one Jordan curve. The notion of surface used was that of a Fréchet surface (see, for instance, [**38**, pp. 350–352]) with specified topological type. In the early 1950's an *existence theorem* was proved independently by Cesari [**6**], Danskin [**8**], and Sigalov [**54**] for essentially the most general type of parametric problem with $\nu = 2$ using Fréchet surfaces. In all these cases, use of conformal maps of the surfaces essentially reduced the problem to a nonparametric one; the area integral was replaced by the Dirichlet integral, since

$$\left(\frac{E+G}{2}\right)^2 = \left(\frac{E-G}{2}\right)^2 + F^2 + (EG - F^2).$$

The nonexistence of special maps of higher dimensional manifolds held up the solution of the problem of Plateau in higher dimensions until that of Reifenberg in 1960 [**51**], [**52**], [**53**]. He showed the existence of a compact *point set* of minimum Hausdorff ν-measure among all such in R^N which are bounded in a certain topological sense by a given compact set and showed that the resulting set consisted of the union of an open analytic manifold and a locally compact set of ν-Hausdorff measure zero; in general, this singular set will not be empty since any analytic set is minimal. Very recently, Almgren [2] extended Reifenberg's results to a wide class of parametric problems in higher dimensions using the notions of currents, developed recently by Federer and Fleming [12] and his own notion of a varifold.

Also during the last five years, a theory of "monotone" operators has been developed and generalized by Minty, Browder, Visik, Lions, Leray, and others to yield existence theorems for (weak) solutions of quite general elliptic systems of the form

$$(21) \qquad \int_G \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i} D^{\alpha}\zeta^i \cdot A_i^{\alpha}[x, Dz(x)]dx = 0 \quad \forall \zeta^i \in C_c^{m_i}(G).$$

About a year ago, the author's paper *Partial regularity results for non-Linear elliptic systems* appeared in the Journal of Mathematics and Mechanics [39]. In that paper, I showed that *if each* $z^i \in H_k^{m_i}(G)$ *for some* $k>1$ *and* $m_i \geq 1$, *and if* z *is a solution of* (21), *then* $G = D \cup Z$, *where* D *is open and* Z *is locally compact and of measure zero and each* $z^i \in C^{m_i+2}(D)$ *provided the* A_i^{α} *satisfy the conditions below. Additional differentiability of the* z^i *follows from that of the* A_i^{α}.

The methods used in the proof of that result were very different from those illustrated above and are too technical to present here. The principal new method was an adaptation of one due to Almgren. We assume that the $A_i^{\alpha}(x, p)$ ($p = \{p_{\beta}^j\}$ where β is a multi-index with $0 \leq |\beta| \leq m_j$) are of class C_{μ}^2 for all (x, p) and satisfy the following conditions:

$$|A(x, p)|, \; |A_x(x, p)| \leq MV^{k-1}, \qquad |A_p|, \; |A_{px}|, \; |A_{pp}| \leq MV^{k-2}$$

$$\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} A_{ip_{\beta}^j}^{\alpha}(x, p)\pi_{\alpha}^i \pi_{\beta}^j \geq m^* V^{k-2} |\pi|^2, \quad m^* > 0,$$

$$V^2 = 1 + \sum_i \sum_{|\alpha| \leq m_i} (z_{,\alpha}^i)^2, \qquad k \geq 2.$$

I and a student have extended these results slightly.

Also, about a year ago, De Giorgi produced an example to show that the De Giorgi-Nash-Moser results do not carry over to the cases $N>1$ and E. Giusti and M. Miranda [16] exhibited an *analytic variational problem* in which $N \geq 3$ and $\nu \geq 3$ and in which $z = |x|^{-1}x$ is the unique solution having those boundary values on $\partial B(0, 1)$. Thus it appears that, in general, singularities are to be expected in the solutions of variational problems when $N>1$. However, J. Neças [43],

[44] has proved regularity for solutions of certain systems including some of higher order, in cases where $\nu=2$ and k and N arbitrary. There is also a paper by E. Giusti [15] in which optimal regularity holds except on a singular set of $(\nu-1)$ dimension zero. It would seem that a continuation of research in regularity theory will have to consist in proving theorems about the singular set; this would appear to require different techniques from those used until now.

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